Brane Dynamics for Treatment of Cosmic Strings and Vortons

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Abstract: This course provides a self contained introduction to the general theory of relativistic brane models, of the category that includes point particle, string, and membrane representations for phenomena that can be considered as being confined to a worldsheet of the corresponding dimension (respectively one, two, and three) in a thin limit approximation. The first part of the course is concerned with purely kinematic aspects: it is shown how, to second differential order, the geometry (and in particular the inner and outer curvature) of a brane worldsheet of arbitrary dimension is describable in terms of the first, second, and third fundamental tensor; the extension to a foliation by a congruence of such worldsheets is also briefly discussed. In the next part, it is shown how – to lowest order in the thin limit – the evolution of such a brane worldsheet will always be governed by a simple tensorial equation of motion whose left hand side is the contraction of the relevant surface stress tensor $\overline{T}^{\mu\nu}$ with the (geometrically defined) second fundamental tensor $K_{\mu\nu}{}^{\rho}$, while the right hand side will simply vanish in the case of free motion and will otherwise be just the orthogonal projection of any external force density that may happen to act on the brane. (Allowance for first order deviations from such a thin limit treatment would require evolution equations of a more complicated kind of which a prototype example is presented.) The last part of the course concentrates on the case of a string: in a four dimensional spacetime background, this is only case that is non trivial in the strong sense of having a worldsheet with both dimension and codimension greater than one. This case is of particular importance for cosmology because it applies to the cosmic strings that arise as vortex defects of the vacuum in many kinds of field theory, and that may conceivably play an important role in the evolution of the universe. It is shown how to set up a Lagrangian providing the complete system of (both internal and external) string evolution equations governing the large scale motion of vacuum vortices of both the simple Nielsen-Olesen-Kibble type and of Witten's superconducting type, including allowance for conservative external forces of both electromagnetic and Kalb-Ramond axion type, but neglecting the dissipative effects that are likely to be important soon after string formation but not later

on. The last part of the course deals with closed loops under typical circumstances in which not only dissipative but also conservative (electromagnetic or axion type) external forces can be neglected in a lowest order approximation, so that the evolution equations will be homogeneous, and thus easily soluble both for dynamic circular configurations and also for the stationary (centrifugally supported) configurations known as vortons, which, if they are sufficiently stable, may be of considerable cosmological significance.

As an aide memoire for readers equiped for viewing colour, warmer hues will be used for physical quantities of a material nature, as contrasted with essentially geometric quantities for which cooler hues will be reserved. Among the latter scalar and tensorial fields determined just by the background spacetime geometry itself will be indivated by *blue symbols* while background coordinate or frame dependent quantities will be indicated by *black symbols*. Similarly scalar and tensorial fields determined on the brane worldsheet just by the embedding geometry will be indicated by *green symbols* quantities that are fully determined as scalars or tensors just by the embedding geometry, while internal coordinate or frame dependent quantities on the brane worldsheet will be indicated by *sky-blue symbols*. As far as more essentially physical quantities are concerned, *pink symbols* will be used for background gauge fields while *magenta symbols* will be used for physical background fields. On the other hand *beige symbols* will be used for worldsheet supported gauge fields while *brown symbols* will be used for physical quantities confined to and intrinsic with respect to the worldsheet, and *violet symbols* will be used for quantities of mixed type, including physical fields confined to the worldsheet but defined with respect to an external reference system, as well as external fields that may be defined with respect to fields on the world sheet. Finally *red symbols* will be used for global integrals and coupling constants.

CONTENTS

*Optional (non-essential) branch subsections marked *.*

0.1 Introduction

The present course is an updated and extended version of lectures written up a couple of years ago for a summer school on "Formation and Interactions of Topological Defects" [1] at the Newton Institute in Cambridge. In preparation for the more specific study of strings in the later sections, the first part of this course is intended as an introduction to the systematic study, in a classical relativistic framework, of "branes", meaning physical models in which the relevant fields are confined to supporting worldsheets of lower dimension than the background spacetime. While not entirely new [2], [3], this subject is still at a rather early stage of development (compared with the corresponding quantum theory [4] which has been stimulated by the rise of "superstring theory"), the main motivation for recent work [5] on classical relativistic brane theory being its application to vacuum defects produced by the Kibble mechanism [6], particularly when of composite type as in the case of cosmic strings attached to external domain walls [7] and of cosmic strings carrying internal currents of the kind whose likely existence was first proposed by Witten [8]. A propos of the latter, a noteworthy example of the progress that has been achieved since the previous version [2] of these notes was written is the construction (see Subsection 5.7) for the first time of a class of reasonably simple *analytically explicit* string models [9] that are capable of describing the macroscopic behaviour of such Witten vortices in a manner that is qualitatively satisfactory and quantitatively adequate not just when the current is very small but even for the largest physically admissable values. This makes it possible to provide an analytic description (see Subsection 6.5) of the dynamics of the circular ring configurations whose equilibrium states (see Subsection 6.6) are the simplest examples of what are known as vortons.

Before the presentation in Section 2 of the dynamic laws governing the evolution of a brane worldsheet, Section 1 provides a recapitulation of the essential differential geometric machinery [10], [11] needed for the analysis of a timelike worldsheet of dimension d say in a background space time manifold of dimension n. At this stage no restriction will be imposed on the curvature of the metric – which will as usual be represented with respect to local background coordinates x^μ (μ= 0, ..., n–1) by its components $g_{\mu\nu}$ – though it will be postulated to be flat, or at least stationary or conformally flat, in many of the applications to be discussed later.

1 Worldsheet curvature analysis

1.1 The first fundamental tensor

The development of geometrical intuition and of computationally efficient methods for use in string and membrane theory has been hampered by a tradition of publishing results in untidy, highly gauge dependent, notation (one of the causes being the undue influence still exercised by Eisenhart's obsolete treatise "Riemannian Geometry" [12]). For the intermediate steps in particular calculations

it is of course frequently useful and often indispensible to introduce specifically adapted auxiliary structures, such as curvilinear worldsheet coordinates σ^i (i= 0, ..., d–1) and the associated bitensorial derivatives

$$x^{\mu}{}_{,i} = \frac{\partial x^{\mu}}{\partial \sigma^{i}} , \tag{1}$$

or specially adapted orthonormal frame vectors, consisting of an internal subset of vectors $\iota_{A}{}^{\mu}$ (A= 0, ... , d–1) tangential to the worldsheet and an external subset of vectors $\lambda_{X}{}^{\mu}$ (X = 1, ... , n–d) orthogonal to the worldsheet, as characterised by

$$\iota_{A}{}^{\mu}\iota_{B\mu} = \eta_{AB} , \qquad \iota_{A}{}^{\mu}\lambda_{X\mu} = 0 , \qquad \lambda_{X}{}^{\mu}\lambda_{Y\mu} = \delta_{XY} , \tag{2}$$

where η_{AB} is a fixed d-dimensional Minkowski metric and the Kronecker matrix δ_{XY} is a fixed (n–d)-dimensional Cartesion metric. Even in the most recent literature there are still (under Eisenhart's uninspiring influence) many examples of insufficient effort to sort out the messy clutter of indices of different kinds (Greek or Latin, early or late, small or capital) that arise in this way by grouping the various contributions into simple tensorially covariant combinations. Another inconvenient feature of many publications is that results have been left in a form that depends on some particular gauge choice (such as the conformal gauge for internal string coordinates) which obscures the relationship with other results concerning the same system but in a different gauge.

The strategy adopted here [13] aims at minimising such problems (they can never be entirely eliminated) by working as far as possible with a single kind of tensor index, which must of course be the one that is most fundamental, namely that of the background coordinates, x^{μ}. Thus, to avoid dependence on the internal frame index A (which is lowered and raised by contraction with the fixed d-dimensional Minkowski metric η_{AB} and its inverse η^{AB}) and on the external frame index X (which is lowered and raised by contraction with the fixed (n-d)-dimensional Cartesian metric δ_{XY} and its inverse δ^{XY}), the separate internal frame vectors $\iota_{A}{}^{\mu}$ and external frame vectors $\lambda_{X}{}^{\mu}$ will as far as possible be eliminated in favour of the frame gauge independent combinations

$$\eta^{\mu}_{\ \nu} = \iota_{A}{}^{\mu}\iota^{A}{}_{\nu} , \qquad \perp^{\mu}_{\nu} = \lambda_{X}{}^{\mu}\lambda^{X}{}_{\nu} , \tag{3}$$

of which the former, $\eta^{\mu}_{\ \nu}$, is what will be referred to as the (first) *fundamental tensor* of the metric, which acts as the (rank d) operator of tangential projection onto the world sheet, while the latter, $\perp^{\mu}_{\nu}$, is the complementary (rank n–d) operator of projection orthogonal to the world sheet.

The same principle applies to the avoidance of unnecessary involvement of the internal coordinate indices which are lowered and raised by contraction with the induced metric on the worldsheet as given by

$$\gamma_{ij} = g_{\mu\nu}x^{\mu}{}_{,i}x^{\nu}{}_{,j} , \tag{4}$$

and with its contravariant inverse γ^{ij}. After being cast (by index raising if necessary) into its contravariant form, any internal coordinate tensor can be directly

projected onto a corresponding background tensor in the manner exemplified by the intrinsic metric itself, which gives

$$\eta^{\mu\nu} = \gamma^{ij} x^{\mu}{}_{,i} x^{\nu}{}_{,j} \,, \tag{5}$$

thus providing an alternative (more direct) prescription for the fundamental tensor that was previously introduced via the use of the internal frame in (3). This approach also provides a direct prescription for the orthogonal projector that was introduced via the use of an external frame in (3) but that is also obtainable immediately from (5) as

$$\perp^{\mu}_{\nu} = g^{\mu}{}_{\nu} - \eta^{\mu}{}_{\nu} \,. \tag{6}$$

As well as having the separate operator properties

$$\eta^{\mu}{}_{\rho}\, \eta^{\rho}{}_{\nu} = \eta^{\mu}{}_{\nu} \,, \qquad \perp^{\mu}{}_{\rho} \perp^{\rho}{}_{\nu} = \perp^{\mu}{}_{\nu} \tag{7}$$

the tensors defined by (5) and (6) will evidently be related by the conditions

$$\eta^{\mu}{}_{\rho} \perp^{\rho}{}_{\nu} \;=\; 0 \;=\; \perp^{\mu}{}_{\rho} \eta^{\rho}{}_{\nu} \,. \tag{8}$$

1.2 The inner and outer curvature tensors

In so far as we are concerned with tensor fields such as the frame vectors whose support is confined to the d-dimensional world sheet, the effect of Riemannian covariant differentation ∇_{μ} along an arbitrary directions on the background spacetime will not be well defined, only the corresponding tangentially projected differentiation operation

$$\overline{\nabla}_{\mu} \stackrel{\text{def}}{=} \eta^{\nu}{}_{\mu} \nabla_{\nu} \,, \tag{9}$$

being meaningful for them, as for instance in the case of a scalar field φ for which the tangentially projected gradient is given in terms of internal coordinate differentiation simply by $\overline{\nabla}^{\mu}\varphi = \gamma^{ij} x^{\mu}{}_{,i}\, \varphi_{,ji}$.

An irreducible basis for the various possible covariant derivatives of the frame vectors consists of the *internal rotation* pseudo-tensor $\rho_{\mu}{}^{\nu}{}_{\rho}$ and the *external rotation* (or "twist") pseudo-tensor $\varpi_{\mu}{}^{\nu}{}_{\rho}$ as given by

$$\rho_{\mu}{}^{\nu}{}_{\rho} = \eta^{\nu}{}_{\sigma}\, \iota^{A}{}_{\rho} \overline{\nabla}_{\mu}\, \iota_{A}{}^{\sigma} = -\rho_{\mu\rho}{}^{\nu} \,, \qquad \varpi_{\mu}{}^{\nu}{}_{\rho} = \perp^{\nu}_{\sigma} \lambda^{X}{}_{\rho} \overline{\nabla}_{\mu} \lambda_{X}{}^{\sigma} = -\varpi_{\mu\rho}{}^{\nu} \,, \tag{10}$$

together with their *mixed* analogue $K_{\mu\nu}{}^{\rho}$ which is obtainable in a pair of equivalent alternative forms given by

$$K_{\mu\nu}{}^{\rho} = \perp^{\rho}_{\sigma} \iota^{A}{}_{\nu} \overline{\nabla}_{\mu}\, \iota_{A}{}^{\sigma} = -\eta^{\sigma}{}_{\nu}\, \lambda_{X}{}^{\rho} \overline{\nabla}_{\mu} \lambda^{X}{}_{\sigma} \,. \tag{11}$$

The reason for qualifying the fields (10) as "pseudo" tensors is that although they are tensorial in the ordinary sense with respect to changes of the background coordinates x^{μ} they are not geometrically well defined just by the geometry of the world sheet but are gauge dependent in the sense of being functions

of the choice of the internal and external frames $\iota_{_A}{}^\mu$ and $\lambda_{_X}{}^\mu$. The gauge dependence of $\rho_\mu{}^\nu{}_\rho$ and $\varpi_\mu{}^\nu{}_\rho$ means that both of them can be set to zero at any chosen point on the worldsheet by choice of the relevant frames in its vicinity. However the condition for it to be possible to set these pseudo-tensors to zero throughout an open neigbourhood is the vanishing of the curvatures of the corresponding frame bundles as characterised with respect to the respective invariance subgroups SO(1,d–1) and SO(n–d) into which the full Lorentz invariance group SO(1,n–1) is broken by the specification of the d-dimensional world sheet orientation. The *inner curvature* that needs to vanish for it to be possible for $\rho_\mu{}^\nu{}_\rho$ to be set to zero in an open neighbourhood is of Riemannian type, is obtainable (by a calculation of the type originally developed by Cartan that was made familiar to physicists by Yang Mills theory) as [10]

$$R_{\kappa\lambda}{}^\mu{}_\nu = 2\eta^\mu{}_\sigma\eta^\tau{}_\mu\eta^\pi{}_{[\lambda}\overline{\nabla}_{\kappa]}\rho_\pi{}^\sigma{}_\tau + 2\rho_{[\kappa}{}^{\mu\pi}\rho_{\lambda]\pi\nu}\,, \tag{12}$$

while the *outer curvature* that needs to vanish for it to be possible for the "twist" tensor $\varpi_\mu{}^\nu{}_\rho$ to be set to zero in an open neighbourhood is of a less familiar type that is given [10] by

$$\Omega_{\kappa\lambda}{}^\mu{}_\nu = 2\perp^\mu_\sigma\perp^\tau_\mu\eta^\pi{}_{[\lambda}\overline{\nabla}_{\kappa]}\varpi_\pi{}^\sigma{}_\tau + 2\varpi_{[\kappa}{}^{\mu\pi}\varpi_{\lambda]\pi\nu}\,. \tag{13}$$

The frame gauge invariance of the expressions (12) and (13), – which means that $R_{\kappa\lambda}{}^\mu{}_\nu$ and $\Omega_{\kappa\lambda}{}^\mu{}_\nu$ are unambiguously well defined as tensors in the strictest sense of the word – is not immediately obvious from the foregoing formulae, but it is made manifest in the the alternative expressions given in Subsection 1.5.

1.3 The second fundamental tensor

Another, even more fundamentally important, gauge invariance property that is not immediately obvious from the traditional approach – as recapitulated in the preceeding subsection is – that of the entity $K_{\mu\nu}{}^\rho$ defined by the mixed analogue (11) of (10), which (unlike $\rho_\mu{}^\nu{}_\rho$ and $\varpi_\mu{}^\nu{}_\rho$, but like $R_{\kappa\lambda}{}^\mu{}_\nu$ and $\Omega_{\kappa\lambda}{}^\mu{}_\nu$) is in fact a geometrically well defined tensor in the strict sense. To see that the formula (11) does indeed give a result that is frame gauge independent, it suffices to verify that it agrees with the alternative – manifestly gauge independent definition [5]

$$K_{\mu\nu}{}^\rho \stackrel{\rm def}{=} \eta^\sigma{}_\nu\overline{\nabla}_\mu\eta^\rho{}_\sigma\,. \tag{14}$$

whereby the entity that we refer to as the *second fundamental tensor* is constructed directly from the the first fundamental tensor $\eta^{\mu\nu}$ as given by (5).

Since this second fundamental tensor, $K_{\mu\nu}{}^\rho$ will play a very important role throughout the work that follows, it is worthwhile to linger over its essential properties. To start with it is to be noticed that a formula of the form (14) could of course be meaningfully meaningful applied not only to the fundamental projection tensor of a d-surface, but also to any (smooth) field of rank-d projection operators $\eta^\mu{}_\nu$ as specified by a field of arbitrarily orientated d-surface elements.

What distinguishes the integrable case, i.e. that in which the elements mesh together to form a well defined d-surface through the point under consideration, is the condition that the tensor defined by (14) should also satisfy the *Weingarten identity*

$$K_{[\mu\nu]}{}^{\rho} = 0 \tag{15}$$

(where the square brackets denote antisymmetrisation), this symmetry property of the second fundamental tensor being derivable [5], [10] as a version of the well known Frobenius theorem. In addition to this non-trivial symmetry property, the second fundamental tensor is also obviously tangential on the first two indices and almost as obviously orthogonal on the last, i.e.

$$\perp^{\sigma}_{\mu} K_{\sigma\nu}{}^{\rho} = K_{\mu\nu}{}^{\sigma}\eta_{\sigma}{}^{\rho} = 0\,. \tag{16}$$

The second fundamental tensor $K_{\mu\nu}{}^{\rho}$ has the property of fully determining the tangential derivatives of the first fundamental tensor $\eta^{\mu}{}_{\nu}$ by the formula

$$\overline{\nabla}_{\mu}\eta_{\nu\rho} = 2K_{\mu(\nu\rho)} \tag{17}$$

(using round brackets to denote symmetrisation) and it can be seen to be characterisable by the condition that the orthogonal projection of the acceleration of any tangential unit vector field u^{μ} will be given by

$$u^{\mu}u^{\nu}K_{\mu\nu}{}^{\rho} = \perp^{\rho}_{\mu}\dot{u}^{\mu}\,, \qquad \dot{u}^{\mu} = u^{\nu}\nabla_{\nu}u^{\mu}\,. \tag{18}$$

1.4 The extrinsic curvature vector and the conformation tensor

It is very practical for a great many purposes to introduce the *extrinsic curvature vector* K^{μ}, defined as the trace of the second fundamental tensor, which is automatically orthogonal to the worldsheet,

$$K^{\mu} \overset{\text{def}}{=} K^{\nu}{}_{\nu}{}^{\mu}\,, \qquad \eta^{\mu}{}_{\nu}K^{\nu} = 0\,. \tag{19}$$

It is useful for many specific purposes to work this out in terms of the intrinsic metric γ_{ij} and its determinant $|\gamma|$. It suffices to use the simple expression $\overline{\nabla}^{\mu}\varphi = \gamma^{ij}x^{\mu}{}_{,i}\varphi_{,j}$ for the tangentially projected gradient of a scalar field φ on the worldsheet, but for a tensorial field (unless one is using Minkowski coordinates in a flat spacetime) there will also be contributions involving the background Riemann Christoffel connection

$$\Gamma_{\mu}{}^{\nu}{}_{\rho} = g^{\nu\sigma}\big(g_{\sigma(\mu,\rho)} - \tfrac{1}{2}g_{\mu\rho,\sigma}\big)\,. \tag{20}$$

The curvature vector is thus obtained in explicit detail as

$$K^{\nu} = \overline{\nabla}_{\mu}\eta^{\mu\nu} = \frac{1}{\sqrt{\|\gamma\|}}\Big(\sqrt{\|\gamma\|}\gamma^{ij}x^{\nu}{}_{,i}\Big)_{,j} + \gamma^{ij}x^{\mu}{}_{,i}x^{\rho}{}_{,j}\Gamma_{\mu}{}^{\nu}{}_{\rho}\,. \tag{21}$$

This last expression is technically useful for certain specific computational purposes, but it must be remarked that much of the literature on cosmic string dynamics has been made unnecessarily heavy to read by a tradition of working all the time with long strings of non tensorial terms such as those on the right of (21) rather than taking advantage of such more succinct tensorial expressions as the preceeding formula $\overline{\nabla}_\mu \eta^{\mu\nu}$. As an alternative to the universally applicable tensorial approach advocated here, there is of course another more commonly used method of achieving succinctness in particular circumstances, which is to sacrifice gauge covariance by using specialised kinds of coordinate system. In particular for the case of a string, i.e. for a 2-dimensional worldsheet, it is standard practise to use conformal coordinates σ^0 and σ^1 so that the corresponding tangent vectors $\dot{x}^\mu = x^\mu_{\ ,0}$ and $x'^\mu = x^\mu_{\ ,1}$ satisfy the restrictions $\dot{x}^\mu x'_{\ \mu} = 0$, $\dot{x}^\mu \dot{x}_\mu + x'^\mu x'_{\ \mu} = 0$, which implies $\sqrt{\Vert\gamma\Vert} = x'^\mu x'_{\ \mu} = -\dot{x}^\mu \dot{x}_\mu$ so that (21) simply gives $\sqrt{\Vert\gamma\Vert}\, K^\nu = x''^\nu - \ddot{x}^\nu + (x'^\mu x'^\rho - \dot{x}^\mu \dot{x}^\rho)\Gamma_{\mu\ \rho}^{\ \nu}$.

The physical specification of the extrinsic curvature vector (19) for a timelike d-surface in a dynamic theory provides what can be taken as the equations of extrinsic motion of the d-surface [5], [11], the simplest possibility being the "harmonic" condition $K^\mu = 0$ that is obtained (as will be shown in the following sections) from a surface measure variational principle such as that of the Dirac membrane model [2], or of the Goto-Nambu string model [6] whose dynamic equations in a flat background are therefore expressible with respect to a standard conformal gauge in the familiar form $x''^\mu - \ddot{x}^\mu = 0$.

There is a certain analogy between the Einstein vacuum equations, which impose the vanishing of the trace $\mathcal{R}_{\mu\nu}$ of the background spacetime curvature $\mathcal{R}_{\lambda\mu}{}^\rho{}_\nu$, and the Dirac-Gotu-Nambu equations, which impose the vanishing of the trace K^ν of the second fundamental tensor $K_{\lambda\mu}{}^\nu$. Just as it is useful to separate out the Weyl tensor [14], i.e. the trace free part of the Ricci background curvature which is the only part that remains when the Einstein vacuum equations are satisfied, so also analogously, it is useful to separate out the the trace free part of the second fundamental tensor, namely the extrinsic conformation tensor [10], which is the only part that remains when equations of motion of the Dirac - Goto - Nambu type are satisfied. Explicitly, the trace free *extrinsic conformation* tensor $C_{\mu\nu}{}^\rho$ of a d-dimensional imbedding is defined [10] in terms of the corresponding first and second fundamental tensors $\eta_{\mu\nu}$ and $K_{\mu\nu}{}^\rho$ as

$$C_{\mu\nu}{}^\rho \stackrel{\rm def}{=} K_{\mu\nu}{}^\rho - \frac{1}{\rm d}\eta_{\mu\nu} K^\rho \ , \qquad C^\nu{}_\nu{}^\mu = 0 \ . \tag{22}$$

Like the Weyl tensor $\mathcal{W}_{\lambda\mu}{}^\rho{}_\nu$ of the background metric (whose definition is given implicitly by (27) below) this conformation tensor has the noteworthy property of being invariant with respect to conformal modifications of the background metric:

$$g_{\mu\nu} \mapsto {\rm e}^{2\alpha} g_{\mu\nu} \ , \qquad \Rightarrow \qquad K_{\mu\nu}{}^\rho \mapsto K_{\mu\nu}{}^\rho + \eta_{\mu\nu} \perp^{\rho\sigma} \nabla_\sigma \alpha \ , \qquad C_{\mu\nu}{}^\rho \mapsto C_{\mu\nu}{}^\rho \ . \tag{23}$$

This formula is useful [40] for calculations of the kind undertaken by Vilenkin [38] in a standard Robertson-Walker type cosmological background, which can

be obtained from a flat auxiliary spacetime metric by a conformal transformation for which e^{α} is a time dependent Hubble expansion factor.

1.5 *The Codazzi, Gauss, and Schouten identities

As the higher order analogue of (14) we can go on to introduce the *third* fundamental tensor[5] as

$$\Xi_{\lambda\mu\nu}{}^{\rho} \stackrel{\rm def}{=} \eta^{\sigma}{}_{\mu}\eta^{\tau}{}_{\nu}\perp^{\rho}_{\alpha}\overline{\nabla}_{\lambda}K_{\sigma\tau}{}^{\alpha}\,, \tag{24}$$

which by construction is obviously symmetric between the second and third indices and tangential on all the first three indices. In a spacetime background that is flat (or of constant curvature as is the case for the DeSitter universe model) this third fundamental tensor is fully symmetric over all the first three indices by what is interpretable as the *generalised Codazzi identity* which is expressible [10] in a background with arbitrary Riemann curvature $\mathcal{R}_{\lambda\mu}{}^{\rho}{}_{\sigma}$ as

$$\Xi_{\lambda\mu\nu}{}^{\rho} = \Xi_{(\lambda\mu\nu)}{}^{\rho} + \tfrac{2}{3}\eta^{\sigma}{}_{\lambda}\eta^{\tau}{}_{(\mu}\eta^{\alpha}{}_{\nu)}\mathcal{R}_{\sigma\tau}{}^{\beta}{}_{\alpha}\perp^{\rho}_{\beta} \tag{25}$$

It is to be noted that a script symbol $\mathcal{R}$ is used here in order to distinguish the (n - dimensional) background Riemann curvature tensor from the intrinsic curvature tensor (12) of the (d - dimensional) worldship to which the ordinary symbol R has already allocated.

For many of the applications that will follow it will be sufficient just to treat the background spacetime as flat, i.e. to take $\mathcal{R}_{\sigma\tau}{}^{\beta}{}_{\alpha}=0$. At this stage however, we shall allow for an unrestricted background curvature. For n$>$ 2 this will be decomposible in terms of its trace free Weyl part $\mathcal{W}_{\mu\nu}{}^{\rho}{}_{\sigma}$ (which as remarked above is conformally invariant) and the corresponding background Ricci tensor and its scalar trace,

$$\mathcal{R}_{\mu\nu} = \mathcal{R}_{\rho\mu}{}^{\rho}{}_{\nu}\,, \qquad \mathcal{R} = \mathcal{R}^{\nu}{}_{\nu}\,, \tag{26}$$

in the form [14]

$$\mathcal{R}_{\mu\nu}{}^{\rho\sigma} = \mathcal{W}_{\mu\nu}{}^{\rho\sigma} + \tfrac{4}{n-2}g^{[\rho}{}_{[\mu}\mathcal{R}^{\sigma]}{}_{\nu]} - \tfrac{2}{(n-1)(n-2)}\mathcal{R}g^{[\rho}{}_{[\mu}g^{\sigma]}{}_{\nu]}\,, \tag{27}$$

(in which the Weyl contribution can be non zero only for n$\geq$ 4). In terms of the tangential projection of this background curvature, one can evaluate the corresponding *internal* curvature tensor (12) in the form

$$R_{\mu\nu}{}^{\rho}{}_{\sigma} = 2K^{\rho}{}_{[\mu}{}^{\tau}K_{\nu]\sigma\tau} + \eta^{\kappa}{}_{\mu}\eta^{\lambda}{}_{\nu}\mathcal{R}_{\kappa\lambda}{}^{\alpha}{}_{\tau}\eta^{\rho}{}_{\alpha}\eta^{\tau}{}_{\sigma}\,, \tag{28}$$

which is the translation into the present scheme of what is well known in other schemes as the *generalised Gauss identity*. The much less well known analogue for the (identically trace free and conformally invariant) *outer* curvature (13) (for which the most historically appropriate name might be argued to be that of Schouten [14]) is given [10] in terms of the corresponding projection of the background Weyl tensor by the expression

$$\Omega_{\mu\nu}{}^{\rho}{}_{\sigma} = 2C_{[\mu}{}^{\tau\rho}C_{\nu]\tau\sigma} + \eta^{\kappa}{}_{\mu}\eta^{\lambda}{}_{\nu}\mathcal{W}_{\kappa\lambda}{}^{\alpha}{}_{\tau}\perp^{\rho}_{\alpha}\perp^{\tau}_{\sigma}\,. \tag{29}$$

It follows from this last identity that in a background that is flat or conformally flat (for which it is necessary, and for n$\geq$ 4 sufficient, that the Weyl tensor should vanish) the vanishing of the extrinsic conformation tensor $C_{\mu\nu}{}^{\rho}$ will be sufficient (independently of the behaviour of the extrinsic curvature vector K^{μ}) for vanishing of the outer curvature tensor $\Omega_{\mu\nu}{}^{\rho}{}_{\sigma}$, which is the condition for it to be possible to construct fields of vectors λ^{μ} orthogonal to the surface and such as to satisfy the generalised Fermi-Walker propagation condition to the effect that $\perp^{\rho}_{\mu}\overline{\nabla}_{\nu}\lambda_{\rho}$ should vanish. It can also be shown [10] (taking special trouble for the case d=3) that in a conformally flat background (of arbitrary dimension n) the vanishing of the conformation tensor $C_{\mu\nu}{}^{\rho}$ is always sufficient (though by no means necessary) for conformal flatness of the induced geometry in the imbedding.

1.6 *Extension to a foliation

It is useful for many purposes to extend the analysis of a single embedded d-surface (as described in the preceeding subsections) to the more general context of a smooth foliation by a congruence of such surfaces. Even when the object of physical interest is just a single embedded surface it is geometrically instructive to consider its relation to neighbouring surfaces of a purely mathematical nature, and of course there are also many applications involving modes in which a foliation by congruence of surface is a physically essential constituent structure (for example the congruence of flow trajectories in an ordinary fluid model [15], or the congruence of string-like vortex lines in the kind of model [16] appropriate for the macroscopic description of a rotating superfluid.)

In the case of such a foliation, the first fundamental tensor $\eta^{\mu\nu}$ as specified by (5) and of the corresponding orthogonal projector $\perp^{\mu}_{\nu}$ as specified by (6) will no longer have support confined to a single d-surface, but will be well defined as ordinary tensor fields over at least an open neighbourhood of the background spacetime. This means that they will have well defined (Riemannian or pseudo-Riemannian) covariant derivatives of the ordinary unrestricted kind (not just tangential covariant derivatives of the kind described in the preceeding sections). These covariant derivatives given will be fully determined by the specification of a certain (first) *deformation tensor*, $\mathcal{H}_{\mu}{}^{\nu}{}_{\rho}$ say, via an expression of the form

$$\nabla_{\mu}\,\eta^{\nu}{}_{\rho} = -\nabla_{\mu}\perp^{\nu}_{\rho} = \mathcal{H}_{\mu}{}^{\nu}{}_{\rho} + \mathcal{H}_{\mu\nu}{}^{\rho}\,. \tag{30}$$

It can easily be seen from (7) that the required deformation tensor will be given simply by

$$\mathcal{H}_{\mu}{}^{\rho}{}_{\nu} = \eta^{\rho}{}_{\sigma}\,\nabla_{\mu}\,\eta^{\sigma}{}_{\nu} = -\perp^{\sigma}_{\nu}\nabla_{\mu}\perp^{\rho}_{\sigma}\,. \tag{31}$$

The middle and last indices of this tensor will evidently have the respective properties of tangentiality and orthogonality that are expressible as

$$\perp^{\sigma}_{\nu}\mathcal{H}_{\mu}{}^{\rho}{}_{\sigma} = 0\,, \qquad\qquad \mathcal{H}_{\mu}{}^{\rho}{}_{\sigma}\,\eta^{\sigma}{}_{\nu} = 0\ . \tag{32}$$

There is no automatic tangentiality or orthogonality property for the first index of the deformation tensor (31), which is thus reducible with respect to the

tangential and orthogonally lateral projections (5)(6) to a sum

$$\mathcal{H}_{\mu}{}^{\rho}{}_{\nu} = K_{\mu}{}^{\rho}{}_{\nu} - L_{\mu\nu}{}^{\rho} \tag{33}$$

in which such a property is obtained for each of the parts

$$K_{\mu}{}^{\rho}{}_{\nu} = \eta^{\sigma}{}_{\mu}\mathcal{H}_{\sigma}{}^{\rho}{}_{\nu}\,, \qquad L_{\mu\nu}{}^{\rho} = -\perp^{\sigma}_{\mu}\mathcal{H}_{\sigma}{}^{\rho}{}_{\nu}\,, \tag{34}$$

which satisfy the conditions

$$\perp^{\sigma}_{\mu}K_{\sigma}{}^{\rho}{}_{\nu} = \perp^{\rho}_{\sigma}K_{\mu}{}^{\sigma}{}_{\nu} = 0 = K_{\mu}{}^{\rho}{}_{\sigma}\,\eta^{\sigma}{}_{\nu}\,. \tag{35}$$

and

$$\eta^{\sigma}{}_{\mu}L_{\sigma}{}^{\rho}{}_{\nu} = \eta^{\rho}{}_{\sigma}L_{\mu}{}^{\sigma}{}_{\nu} = 0 = L_{\mu}{}^{\rho}{}_{\sigma}\perp^{\sigma}_{\nu}\,. \tag{36}$$

The first of these decomposed parts, which might be described as the *tangential turning* tensor, is expressible explicitly as

$$K_{\mu}{}^{\rho}{}_{\nu} = \eta^{\rho}{}_{\sigma}\eta^{\tau}{}_{\mu}\overline{\nabla}_{\tau}\eta^{\sigma}{}_{\nu}\,, \tag{37}$$

and is thus evidently identifiable with the *second fundamental tensor* as defined by (14). The corresponding explicit formula for the second part of the decomposition (33) is

$$L_{\mu}{}^{\rho}{}_{\nu} = \perp^{\rho}_{\sigma}\perp^{\tau}_{\mu}\overline{\nabla}_{\tau}\perp^{\sigma}_{\nu}\,. \tag{38}$$

Unlike $K_{\mu\rho}{}^{\nu}$, which by the Weingarten identity (15) is automatically symmetric on its first two (tangential) indices, the *lateral turning tensor*, $L_{\mu\nu}{}^{\rho}$ will in general have an antisymmetric part, $\omega_{\mu\nu}{}^{\rho}$ say, as well as a symmetric part, $\theta_{\mu\rho}{}^{\nu}$ say, with respect to its first two (orthogonal) indices. It will be decomposible in terms of these parts in the form

$$L_{\mu\nu}{}^{\rho} = \omega_{\mu\nu}{}^{\rho} + \theta_{\mu\nu}{}^{\rho}\,, \tag{39}$$

with

$$\omega_{(\mu\nu)}{}^{\rho} = 0\,, \qquad \theta_{[\mu\nu]}{}^{\rho} = 0\,. \tag{40}$$

Just as the Wiengarten symmetry property (15) of $K_{\mu\nu}{}^{\rho}$ is interpretable as the integrability condition for the d-surfaces under consideration, so by analogy, it is evident that the special condition that is necessary and sufficient for the existence of a complementary orthogonal foliation by (n-d)-surfaces is that $L_{\mu\nu}{}^{\rho}$ should also be symmetric, i.e. that the rotation tensor $\omega_{\mu\nu}{}^{\rho}$ should vanish.

The part of $L_{\mu\nu}{}^{\rho}$ that remains even if the foliation is (n-d) surface orthogonal is the symmetric part, $\theta_{\mu\nu}{}^{\rho}$, which is the natural generalisation of the usual two index divergence tensor $\theta_{\mu\nu}$ of an ordinary fluid flow. For a 1-dimensional timelike foliation, which will have a unique future directed unit tangent vector u^{μ}, the contraction $\theta_{\mu\nu} = \theta_{\mu\nu}{}^{\rho}u_{\rho}$ gives the usual divergence tensor, whose trace $\theta_{\nu}{}^{\nu} = \nabla_{\mu}u^{\mu}$ is the ordinary scalar divergence of the flow. The evolution of this divergence scalar is governed by the Raychaudhuri identity whose generalisation to an evolution equation for $\theta_{\mu\nu}$ was given by Hawking and Ellis [15]. In the next

subsection it will be shown how (following Capovilla and Guven [17]) one can obtain a further extension (74) of the Raychaudhuri identity that provides an evolution equation for the three index generalised divergence tensor $\theta_{\mu\nu}{}^{\rho}$ which (unlike the ordinary divergence tensor $\theta_{\mu\nu}$) is always well defined whatever the dimension of the foliation.

Just as the first order derivatives of the fundamental tensor $\eta^{\mu}{}_{\nu}$ of a foliation are specifiable by the first deformation tensor $\mathcal{H}_{\mu}{}^{\nu}{}_{\rho}$ defined in the previous section, so analogously the derivatives of the next order will be fully determined by the further specification of a corresponding *second deformation tensor*, $\mathcal{X}_{\lambda\mu}{}^{\nu}{}_{\rho}$ say, via an expression of the slightly less simple form

$$\nabla_{\!\lambda}\, \mathcal{H}_{\mu}{}^{\nu}{}_{\rho} = \mathcal{X}_{\lambda\mu}{}^{\nu}{}_{\rho} + \mathcal{H}_{\lambda\sigma}{}^{\nu}\mathcal{H}_{\mu}{}^{\sigma}{}_{\rho} - \mathcal{H}_{\lambda\rho}{}^{\sigma}\mathcal{H}_{\mu}{}^{\nu}{}_{\sigma}\,, \tag{41}$$

that is easily derivable from the definition

$$\mathcal{X}_{\lambda\mu}{}^{\nu}{}_{\rho} = \eta^{\nu}{}_{\sigma}\perp^{\tau}_{\rho}\nabla_{\!\lambda}\, \mathcal{H}_{\mu}{}^{\sigma}{}_{\tau}\,, \tag{42}$$

using the projection properties (32).

Just as the second fundamental tensor of an individual surface was obtained from the first deformation tensor of the foliation by tangential projection according to (33), so analogously the corresponding *third fundamental tensor*, $\Xi_{\lambda\mu}{}^{\nu}{}_{\rho}$, as defined by (24), will be obtainable from the second deformation tensor by the slightly less simple tangential projection operation

$$\Xi_{\lambda\mu}{}^{\nu}{}_{\rho} = \eta^{\alpha}{}_{\lambda}\,\eta^{\beta}{}_{\mu}\big(\mathcal{X}_{\alpha\beta}{}^{\nu}{}_{\rho} + \mathcal{H}_{\alpha\beta}{}^{\sigma}\mathcal{H}_{\sigma}{}^{\nu}{}_{\rho}\big)\,. \tag{43}$$

It can be seen to follow from the Weingarten integrability property (15) that the antisymmetric part of this third fundamental tensor will be given directly by the simple projection operation

$$\Xi_{[\lambda\mu]}{}^{\nu}{}_{\rho} = \eta^{\alpha}{}_{\lambda}\,\eta^{\beta}{}_{\mu}\mathcal{X}_{[\alpha\beta]}{}^{\nu}{}_{\rho}\,. \tag{44}$$

It is easy to use the projection properties (32) to verify that the tensor obtained by this procedure does indeed satisfy the original definition (24) of the third fundamental tensor, which can thereby be seen to be unambigously determined, like the second fundamental tensor, just by the specification of an individual imbedded surface, independently of any extension to a space covering foliation.

The higher order differential information contained in the second deformation tensor and the third fundamental tensor is not needed for the evaluation of the curvature tensors discussed in the following sections, even though they are dependent on the second derivatives of the background metric itself. However a useful biproduct of the curvature analysis that follows is the derivation of of a separation identity (interpretable as a generalisation of the historic Codazzi identity) that expresses the antisymmetric part of the second deformation tensor – and hence by (44) that of the third fundamental tensor – directly as the corresponding projection of the background spacetime curvature $\mathcal{R}_{\lambda\mu}{}^{\nu}{}_{\rho}$.

1.7 *The adapted connection and curvature of a foliation.

Due to existence of the decomposition whereby a background spacetime vector, with components ξ^μ say, is split up as the sum of its surface tangential part $\eta^\mu_{\ \nu}\,\xi^\nu$ and its surface orthogonal part $\perp^\mu_{\nu}\xi^\nu$, there will be a corresponding adaptation of the ordinary concept of parallel propagation with respect to the background connection $\Gamma_{\mu\ \rho}^{\ \nu}$. The principle of the adapted propagation concept is to follow up an ordinary operation of infinitesimal parallel propagation by the projection adjustment that is needed to ensure that purely tangential vectors propagate onto purely tangential vectors while purely orthogonal vectors propagate onto purely orthogonal vectors. Thus for purely tangential vectors, the effect of the adapted propagation is equivalent to that of ordinary internal parallel propagation with respect to the induced metric in the imbedded surface, while for purely orthogonal vectors it is interpretable as the natural generalisation of the standard concept of Fermi-Walker propagation. For an infinitesimal displacement dx^μ the deviation between the actual component variation $(dx^\nu)\partial_\nu\xi^\mu$ and the variation that would be obtained by the corresponding adapted propagation law will be expressible in the form $(dx^\nu)\mathcal{D}_\nu\xi^\mu$ where $\mathcal{D}$ denotes the corresponding *adapted differentiation* operator, whose effect will evidently be given by

$$\mathcal{D}_\nu\,\xi^\mu = \eta^\mu_{\ \rho}\nabla_\mu\big(\eta^\rho_{\ \sigma}\xi^\sigma\big) + \perp^\mu_\rho\nabla_\mu\big(\perp^\rho_\sigma\xi^\sigma\big)\,. \tag{45}$$

It can thus be seen that this operation will be expressible in the form

$$\mathcal{D}_\nu\,\xi^\mu = \nabla_\mu\xi^\nu + \alpha_{\nu\ \sigma}^{\ \mu}\xi^\sigma = \partial_\nu\,\xi^\nu + \mathcal{A}_{\nu\ \sigma}^{\ \mu}\xi^\sigma\,, \tag{46}$$

where the adapted *foliation connection* components $\mathcal{A}_{\mu\ \rho}^{\ \nu}$ are given by the formula

$$\mathcal{A}_{\mu\ \rho}^{\ \nu} = \Gamma_{\mu\ \rho}^{\ \nu} + \alpha_{\mu\ \rho}^{\ \nu}\,, \tag{47}$$

in which the $\alpha_{\mu\ \rho}^{\ \nu}$ are the components of the relevant *adaptation tensor*, whose components can be seen from (45) to be given by

$$\alpha_{\mu\ \rho}^{\ \nu} = \big(\eta^\nu_{\ \sigma} - \perp^\nu_\sigma\big)\nabla_\mu\,\eta^\sigma_{\ \rho}\,. \tag{48}$$

The entirely covariant index lowered version of this adaptation tensor can thus be seen to be expressible in terms of the deformation tensor defined by (31) in the form

$$\alpha_{\mu\nu\rho} = 2\mathcal{H}_{\mu[\nu\rho]}\,. \tag{49}$$

The fact that this last expression is manifestly antisymmetric with respect to the last two indices of the adaptation tensor makes it evident that, like the usual Riemannian differentiation operator ∇, the adapted differentiation operator $\mathcal{D}$ will commute with index raising or lowering, since the metric remains invariant under adapted propagation:

$$\mathcal{D}_\mu\,g_{\nu\rho} = 0\,. \tag{50}$$

However, unlike ∇, the adapted differentiation operator has the convenient property of also commuting with tangential and orthogonal projection, since it can be

seen to follow from (30) and (32) that the corresponding operators also remain invariant under adapted propagation:

$$\mathcal{D}_\mu\, \eta^\nu{}_\rho = 0\ , \qquad \mathcal{D}_\mu \perp^{\!\nu}{}_\rho = 0\,. \tag{51}$$

There is of course a price to be paid in order to obtain this considerable advantage of $\mathcal{D}$ over ∇, but it is not exhorbitant: all that has to be sacrificed is the analogue of the symmetry property

$$\Gamma_{[\mu}{}^\nu{}_{\rho]} = 0\,. \tag{52}$$

expressing the absence of torsion in the Riemannian case. For the adapted connection $\mathcal{A}_\mu{}^\nu{}_\rho$, the torsion tensor defined by

$$\Theta_\mu{}^\nu{}_\rho = 2\mathcal{A}_{[\mu}{}^\nu{}_{\rho]} = 2\alpha_{[\mu}{}^\nu{}_{\rho]}\ , \tag{53}$$

will not in general be zero.

When one is dealing not with an extended foliation but just with an individual imbedded surface, the general adapted differentiation operator $\mathcal{D}$ will not be well defined, but adapted differentiation in tangential directions will still be specified by the corresponding unambiguously defined restricted differentiation operator $\overline{\mathcal{D}}$ whose tensorial component representation is given by the adapted analogue of (9), namely

$$\overline{\mathcal{D}}_\mu \overset{\text{def}}{=} \eta^\nu{}_\mu \mathcal{D}_\nu\,. \tag{54}$$

Restriction to a single imbedded surface will not invalidate the properties of commuting with index raising and tangential or orthogonal projection as expressed in the general case by (50) and (51), so in abstract symbolical notation we shall have not only $\big[\overline{\mathcal{D}}, g\big] = 0$, but also $\big[\overline{\mathcal{D}}, \eta\big] = 0$, and $\big[\overline{\mathcal{D}}, \perp\big] = 0$. It is evident from (46) and(53) that the effect of this tangentially restricted adapted differentiation operator on any surface supported (but not necessarily surface tangential) vector field with components ξ^μ will be given by

$$\overline{\mathcal{D}}_\mu\, \xi^\nu = \overline{\nabla}_\mu \xi^\mu + a_\mu{}^\nu{}_\rho \xi^\rho\ , \qquad a_\mu{}^\nu{}_\rho = \eta^\sigma{}_\mu \alpha_\sigma{}^\nu{}_\rho\,, \tag{55}$$

where the relevant tangentially restricted part of the adaptation tensor, which may naturally described as the *acceleration tensor*, is given explicitly as function only of the ordinary (restricted) second fundamental tensor of the imbedded surface by the simple formula

$$a_{\mu\nu\rho} = 2K_{\mu[\nu\rho]}\,. \tag{56}$$

As a simple illustration, it is to be remarked that in the particular case for which the imbedded surface is a particle world-line (with d=1) having a timelike tangent vector ue^μ satisfying the unit normalisation condition $u^\nu u_\nu = -1$, for which the first and second fundamental tensors will be given respectively by $\eta^{\mu\nu} = -u^\mu u^\nu$ and $K_\mu{}^\nu{}_\rho = u_\mu u^\nu \dot{u}_\rho$ where $\dot{u}^\rho = u^\sigma \nabla_\sigma u^\rho$ are the components of the ordinary acceleration vector, it follows that the corresponding acceleration tensor will be given by $a_{\mu\nu\rho} = 2u_\mu u_{[\nu} \dot{u}_{\rho]}$.

The curvature associated with the adapted connection (46) can be read out from the ensuing commutator formula, which, for an abitrary vector field with components ξ^μ, will take the standard form

$$\mathcal{D}_{[\mu}\mathcal{D}_{\nu]}\xi^\rho = \mathcal{F}_{\mu\nu}{}^\rho{}_\sigma\xi^\sigma - \Theta_\mu{}^\sigma{}_\nu\mathcal{D}_\sigma\xi^\rho\,, \tag{57}$$

in which the torsion tensor components $\Theta_\mu{}^\sigma{}_\nu$ are as defined by (53) while the components $\mathcal{F}_{\mu\nu}{}^\rho{}_\sigma$ are defined by a Yang-Mills type curvature formula of the form

$$\mathcal{F}_{\mu\nu}{}^\rho{}_\sigma = 2\partial_{[\mu}\mathcal{A}_{\nu]}{}^\rho{}_\sigma + 2\mathcal{A}_{[\mu}{}^{\rho\tau}\mathcal{A}_{\nu]\tau\sigma}\,. \tag{58}$$

Although the connection components $\mathcal{A}_\mu{}^\nu{}_\rho$ from which it is constructed are not of tensorial type, the resulting curvature components are of course strictly tensorial. This is made evident by evaluating the components (58) of this *amalgamated foliation curvature* in terms of the background curvature tensor

$$\mathcal{R}_{\mu\nu}{}^\rho{}_\sigma = 2\partial_{[\mu}\Gamma_{\nu]}{}^\rho{}_\sigma + \Gamma_\mu{}^\rho{}_\tau\Gamma_\nu{}^\tau{}_\sigma - \Gamma_\nu{}^\rho{}_\tau\Gamma_\mu{}^\tau{}_\sigma\,, \tag{59}$$

and the adaptation tensor $\alpha_\mu{}^\nu{}_\rho$ given by (49), which gives the manifestly tensorial expression

$$\mathcal{F}_{\mu\nu}{}^\rho{}_\sigma = \mathcal{R}_{\mu\nu}{}^\rho{}_\sigma + 2\nabla_{[\mu}\alpha_{\nu]}{}^\rho{}_\sigma + 2\alpha_{[\mu}{}^{\rho\tau}\alpha_{\nu]\tau\sigma}\,. \tag{60}$$

Although it does not share the full set of symmetries of the Riemann tensor, the foliation curvature obtained in this way will evidently be antisymmetric in both its first and last pairs of indices:

$$\mathcal{F}_{\mu\nu\rho\sigma} = \mathcal{F}_{[\mu\nu][\rho\sigma]}\,. \tag{61}$$

Using the formula (49), it can be see from (32) and (41), that the difference between this adapted curvature and the ordinary background Riemann curvature will be given by

$$\mathcal{F}_{\mu\nu}{}^{\rho\sigma} - \mathcal{R}_{\mu\nu}{}^{\rho\sigma} = 4\mathcal{X}_{[\mu\nu]}{}^{[\rho\sigma]} + 2\mathcal{H}_{[\mu}{}^{\sigma\tau}\mathcal{H}_{\nu]}{}^\rho{}_\tau + 2\mathcal{H}_{[\mu}{}^{\tau\rho}\mathcal{H}_{\nu]\tau}{}^\sigma\,, \tag{62}$$

which superficially appears to depend on the higher order derivatives involved in the second deformation tensor $\mathcal{X}_{\mu\nu}{}^{\rho\sigma}$.

This appearance is however deceptive. The reason for qualifying the foliation curvature tensor (58) as "amalgamated" is that because the adapted derivation operator has been constructed in such a way as to map purely tangential vector fields onto purely tangential vector fields, and purely orthogonal vector fields onto purely orthogonal vector fields, it follows that the same applies to the corresponding curvature, which will therefore consist of an additive amalgamation of two separate parts having the form

$$\mathcal{F}_{\mu\nu}{}^\rho{}_\sigma = \mathcal{P}_{\mu\nu}{}^\rho{}_\sigma + \mathcal{Q}_{\mu\nu}{}^\rho{}_\sigma\,, \tag{63}$$

in which the "inner" curvature acting on purely tangential vectors is given by a doubly tangential projection as

$$\mathcal{P}_{\mu\nu}{}^\rho{}_\sigma = \mathcal{F}_{\mu\nu}{}^\kappa{}_\lambda\,\eta^\rho{}_\kappa\,\eta^\lambda{}_\sigma\,, \tag{64}$$

while the "outer" curvature acting on purely orthogonal vectors is given by a doubly othogonal projection as

$$\mathcal{Q}_{\mu\nu}{}^{\rho}{}_{\sigma} = \mathcal{F}_{\mu\nu}{}^{\kappa}{}_{\lambda} \perp^{\!\rho}{}_{\kappa} \perp^{\!\lambda}{}_{\sigma} \,. \tag{65}$$

It is implicit in the separation expressed by (63) that the mixed tangential and orthogonal projection of the adapted curvature must vanish:

$$\mathcal{F}_{\mu\nu}{}^{\kappa}{}_{\lambda}\, \eta^{\rho}{}_{\kappa} \perp^{\!\lambda}{}_{\sigma} = 0 \,. \tag{66}$$

To get back, from the extended foliation curvature tensors that have just been introduced, to their antecedent analogues [10] for an individual embedded surface, the first step is to construct the *amalgamated embedding curvature* tensor, $F_{\mu\nu}{}^{\rho}{}_{\sigma}$ say, which will be obtainable from the corresponding amalgamated foliation curvature $\mathcal{F}_{\mu\nu}{}^{\rho}{}_{\sigma}$ by a doubly tangential projection having the form

$$F_{\mu\nu}{}^{\rho}{}_{\sigma} = \eta^{\alpha}_{\ \mu}\, \eta^{\beta}_{\ \nu} \mathcal{F}_{\alpha\beta}{}^{\rho}{}_{\sigma} \,. \tag{67}$$

As did the extended foliation curvature, so also this amalgated embedding curvature will separate as the sum of "inner" and "outer" parts in the form

$$F_{\mu\nu}{}^{\rho}{}_{\sigma} = R_{\mu\nu}{}^{\rho}{}_{\sigma} + \Omega_{\mu\nu}{}^{\rho}{}_{\sigma} \,, \tag{68}$$

in which the "inner" embedding curvature is given, consistently wit (28) by another doubly tangential projection as

$$R_{\mu\nu}{}^{\rho}{}_{\sigma} = F_{\mu\nu}{}^{\kappa}{}_{\lambda}\, \eta^{\rho}{}_{\kappa}\, \eta^{\lambda}{}_{\sigma} = \eta^{\alpha}_{\ \mu} \eta^{\beta}_{\ \nu} \mathcal{P}_{\alpha\beta}{}^{\rho}{}_{\sigma} \,, \tag{69}$$

while the "outer" embedding curvature is given, consistently with (29) by the corresponding doubly lateral projection as

$$\Omega_{\mu\nu}{}^{\rho}{}_{\sigma} = F_{\mu\nu}{}^{\kappa}{}_{\lambda} \perp^{\!\rho}{}_{\kappa} \perp^{\!\lambda}{}_{\sigma} = \eta^{\alpha}_{\ \mu} \eta^{\beta}_{\ \nu} \mathcal{Q}_{\alpha\beta}{}^{\rho}{}_{\sigma} \,. \tag{70}$$

The non-trival separation identity (66) can be considered as a generalisation to the case of foliations of the relation that is itself interpretable as an extended generalisation to higher dimensions of the historic Codazzi equation that was originally formulated in the restricted context of 3-dimensional flat space. It can be seen from (62) that this extended Codazzi identity is expressible as

$$2\mathcal{X}_{[\mu\nu]}{}^{\rho}{}_{\sigma} + \mathcal{R}_{\mu\nu}{}^{\kappa}{}_{\lambda}\, \eta^{\rho}{}_{\kappa} \perp^{\!\lambda}{}_{\sigma} = 0 \,, \tag{71}$$

which shows that the relevant higher derivatives are all determined entirely by the Riemannian background curvature so that no specific knowledge of the second deformation tensor is needed. The formula (62) can also be used to evaluate the "inner" tangential part of the foliation curvature tensor as

$$\mathcal{P}_{\mu\nu}{}^{\rho}{}_{\sigma} = 2\mathcal{H}_{[\nu}{}^{\rho\tau} \mathcal{H}_{\mu]\sigma\tau} + \mathcal{R}_{\mu\nu}{}^{\kappa}{}_{\lambda}\, \eta^{\rho}{}_{\kappa}\, \eta^{\lambda}{}_{\sigma} \tag{72}$$

and to evaluate the "outer" orthogonal part of the foliation curvature tensor as

$$\mathcal{Q}_{\mu\nu}{}^{\rho}{}_{\sigma} = 2\mathcal{H}_{[\mu}{}^{\tau\rho} \mathcal{H}_{\nu]\tau\sigma} \mathcal{R}_{\mu\nu}{}^{\kappa}{}_{\lambda} \perp^{\!\rho}{}_{\kappa} \perp^{\!\lambda}{}_{\sigma} \,. \tag{73}$$

The formula (72) for the "inner" foliation curvature is evidently classifiable as an extension of the preceeding generalisation (28) of the historic Gauss equation, while the formula (73) for the "outer" foliation curvature is an extension the more recently derived [10] generalisation (29) of what has sometimes been referred to as the "Ricci equation" but what would seem more appropriately describable as the *Schouten equation*, with reference to the earliest relevant source with which I am familiar [14], since long after the time of Ricci it was not yet understood even by such a leading geometer as Eisenhart [12]. In much the same way, the non-trivial separation identity (66) provides the relation (71) that can be considered as a generalisation to the case of foliations of the identity (25) that is itself interpretable as an extended generalisation to higher dimensions of the historic Codazzi equation that was originally formulated in the restricted context of 3-dimensional flat space. The corresponding doubly lateral projection of the foliation curvature would provide the analogous result for the orthogonal foliation by (n-d)-surfaces that would exist in the irrotational case for which the lateral turning tensor given by (39) is symmetric. Finally the corresponding mixed tangential and lateral projection of (62) gives an identity that is expressible in terms of foliation adapted differentiation (46) as

$$\perp^{\tau}_{\mu}\mathcal{D}_{\tau}K_{\nu}{}^{\rho}{}_{\sigma} + \eta^{\tau}{}_{\nu}\mathcal{D}_{\tau}L_{\mu\sigma}{}^{\rho} = K_{\nu}{}^{\lambda}{}_{\mu}K_{\lambda}{}^{\rho}{}_{\sigma} + L_{\mu}{}^{\lambda}{}_{\nu}L_{\lambda\sigma}{}^{\rho} + \eta^{\alpha}{}_{\nu}\perp^{\beta}_{\mu}\mathcal{R}_{\alpha\beta}{}^{\kappa}{}_{\lambda}\,\eta^{\rho}{}_{\kappa}\perp^{\lambda}_{\sigma}\,. \tag{74}$$

This last result is interpretable as the translation into the pure background tensorial formalism used here of the recently derived generalisation [17] to higher dimensional foliations of the well known Raychaudhuri equation (whose original scalar version, and its tensorial extension [15], were formulated just for the special case of a foliation by 1-dimensional curves). The complete identity (71) is therefore interpretable as an amalgamated Raychaudhuri-Codazzi identity.

1.8 *The special case of a string worldsheet in 4-dimensions

The application with which we shall mainly be concerned in the following work will be the case d=2 of a string. An orthonormal tangent frame will consist in this case just of a timelike unit vector, $\iota_0{}^{\mu}$, and a spacelike unit vector, $\iota_1{}^{\mu}$, whose exterior product vector is the frame independent antisymmetric unit surface element tensor

$$\mathcal{E}^{\mu\nu} = 2\iota_0{}^{[\mu}\iota_1{}^{\nu]} = 2\big(-|\gamma|\big)^{-1/2}\, x^{[\mu}{}_{,0}\, x^{\nu]}{}_{,1}\,, \tag{75}$$

whose tangential gradient satisfies

$$\overline{\nabla}_{\lambda}\mathcal{E}^{\mu\nu} = -2K_{\lambda\rho}{}^{[\mu}\mathcal{E}^{\nu]\rho}\,. \tag{76}$$

(This is the special d=2 case of what is, as far as I am aware, the only wrongly printed formula in the more complete analysis [10] on which this section is based: the relevant general formula (B9) is valid as printed only for odd d, but needs insertion of a missing sign adjustment factor $(-1)^{\mathrm{d}-1}$ in order to hold for all

d.) In this case the inner rotation pseudo tensor (10) is determined just by a corresponding rotation covector ρ_μ according to the specification

$$\rho_{\lambda\ \nu}^{\ \mu} = \frac{1}{2}\,\mathcal{E}^\mu{}_\nu \rho_\lambda\,, \qquad\qquad \rho_\lambda = \rho_{\lambda\ \nu}^{\ \mu}\mathcal{E}^\nu{}_\mu\,. \tag{77}$$

This can be used to see from (12) that the Ricci scalar,

$$R = R^\nu{}_\nu \qquad\qquad R_{\mu\nu} = R_{\rho\mu}{}^\rho{}_\nu\,, \tag{78}$$

of the 2-dimensional worldsheet will have the well known property of being a pure surface divergence, albeit of a frame gauge dependent quantity:

$$R = \overline{\nabla}_\mu\big(\mathcal{E}^{\mu\nu}\rho_\nu\big)\,. \tag{79}$$

In the specially important case of a string in ordinary 4-dimensional spacetime, i.e. when we have not only d=2 but also n=4, the antisymmetric background measure tensor $\varepsilon^{\lambda\mu\nu\rho}$ can be used to determine a scalar (or more strictly, since its sign is orientation dependent, a pseudo scalar) magnitude Ω for the outer curvature tensor (13) (despite the fact that its traces are identically zero) according to the specification

$$\Omega = \frac{1}{2}\,\Omega_{\lambda\mu\nu\rho}\,\varepsilon^{\lambda\mu\nu\rho}\,. \tag{80}$$

Under these circumstances one can also define a "twist" covector ϖ_μ, that is the outer analogue of ρ_μ, according to the specification

$$\varpi_\nu = \frac{1}{2}\,\varpi_\nu{}^{\mu\lambda}\,\varepsilon_{\lambda\mu\rho\sigma}\,\mathcal{E}^{\rho\sigma}\,. \tag{81}$$

This can be used to deduce from (13) that the outer curvature (pseudo) scalar Ω of a string worldsheet in 4-dimensions has a divergence property of the same kind as that of its more widely known Ricci analogue (79), the corresponding formula being given by

$$\Omega = \overline{\nabla}_\mu\big(\mathcal{E}^{\mu\nu}\varpi_\nu\big)\,. \tag{82}$$

It is to be remarked that for a compact spacelike 2-surface the integral of (76) gives the well known Gauss Bonnet invariant, but that the timelike string worldsheets under consideration here will not be characterised by any such global invariant since they will not be compact (being open in the time direction even for a loop that is closed in the spacial sense). The outer analogue of the Gauss Bonnet invariant that arises from (80) for a spacelike 2-surface has been discussed by Penrose and Rindler [18] but again there is no corresponding global invariant in the necessarily non-compact timelike case of a string worldsheet.

2 Laws of motion for a regular pure brane complex

2.1 Regular and distributional formulations of a brane action

The term p-brane has come into use [4], [19] to describe a dynamic system localised on a timelike support surface of dimension d=p+1, imbedded in a spacetime background of dimension n>p. Thus at the low dimensional extreme one has the example of a zero - brane, meaning what is commonly referred to as a "point particle", and of a 1-brane meaning what is commonly referred to as a "string". At the high dimensional extreme one has the "improper" case of an (n–1)-brane, meaning what is commonly referred to as a "medium" (as exemplified by a simple fluid), and of an (n–2)-brane, meaning what is commonly referred to as a "membrane" (from which the generic term "brane" is derived). A membrane (as exemplified by a cosmological domain wall) has the special feature of being supported by a hypersurface, and so being able to form a boundary between separate background space time regions; this means that a 2-brane has the status of being a membrane in ordinary 4-dimensional spacetime (with $n = 4$) but not in a higher dimensional (e.g. Kaluza Klein type) background.

The purpose of the present section is to consider the dynamics not just of an individual brane but of a *brane complex* or "rigging model" [5] such as is illustrated by the nautical archetype in which the wind – a 3-brane – acts on a boat's sail – a 2-brane – that is held in place by cords – 1-branes – which meet at knots, shackles and pulley blocks that are macroscopically describable as point particles – i.e. 0-branes. In order for a a set of branes of diverse dimensions to qualify as a"geometrically regular" brane complex or "rigging system" it is required not only that the support surface of each (d–1)-brane should be a smoothly imbedded d-dimensional timelike hypersurface but also that its boundary, if any, should consist of a disjoint union of support surfaces of an attatched subset of lower dimensional branes of the complex. (For example in order qualify as part of a regular brane complex the edge of a boat's sail can not be allowed to flap freely but must be attatched to a hem cord belonging to the complex.) For the brane complex to qualify as regular in the strong dynamic sense that will be postulated in the present work, it is also required that a member p-brane can exert a direct force only on an an attached (p–1)-brane on its boundary or on an attached (p+1)-brane on whose boundary it is itself located, though it may be passively subject to forces exerted by a higher dimensional background field. For instance the Peccei-Quin axion model gives rise to field configurations representable as regular complexes of domain walls attached to strings [7], [20], [21], and a bounded (topological or other) Higgs vortex defect terminated by a pair of pole defects [22], [23], [24], [25], [26], [27] may be represented as a regular brane complex consisting of a finite cosmic string with a pair of point particles at its ends, in an approximation neglecting Higgs field radiation. (However allowance for radiation would require the use of an extended complex including the Higgs medium whose interaction with the string – and a fortiori with the terminating particles – would violate the regularity condition: the ensuing singularities in the back reaction would need

to be treated by a renormalisation procedure of a kind [21], [28], [29], [30] whose development so far has been beset with difficulties in preserving exact local Lorentz invariance, an awkward problem that is beyond the scope of the present article.)

The present section will be restricted to the case of a brane complex that is not only regular in the sense of the preceeding paragraph but that is also *pure* (or "fine") in the sense that the lengthscales characterising the internal structure of the (defect or other) localised phenomenon represented by the brane models are short compared with those characterising the macroscopic variations under consideration so that polarisation effects play no role. For instance in the case of a point particle, the restriction that it should be describable as a "pure" zero brane simply means that it can be represented as a simple monopole without any dipole or higher multipole effects. In the case of a cosmic string the use of a "pure" 1-brane description requires that the underlying vortex defect be sufficiently thin compared not only compared with its total length but also compared with the lengthscales characterising its curvature and the gradients of any currents it may be carrying. The effect of the simplest kind of curvature corrections beyond this "pure brane" limit will be discussed in Subsection 3.5, but in the rest of these lectures, as in the present section, it will be assumed that the ratio of microscopic to macroscopic lengthscales is sufficiently small for description in terms of "pure" p-branes to be adequate.

The present section will not be concerned with the specific details of particular cases but with the generally valid laws that can be derived as Noether identities from the postulate that the model is governed by dynamical laws derivable from a variational principle specified in terms of an action function $\mathcal{I}$. It is however to be emphasised that the validity at a macroscopic level of the laws given here is not restricted to cases represented by macroscopic models of the strictly conservative type directly governed by a macroscopic variational principle. The laws obtained here will also be applicable to classical models of dissipative type (e.g. allowing for resistivity to relative flow by internal currents) as necessary conditions for the existence of an underlying variational description of the microscopic (quantum) degrees of freedom that are allowed for merely as entropy in the macroscopically averaged classical description.

In the case of a brane complex, the total action $\mathcal{I}$ will be given as a sum of distinct d-surface integrals respectively contributed by the various (d–1)-branes of the complex, of which each is supposed to have its own corresponding Lagrangian surface density scalar ${}^{(\rm d)}\overline{\mathcal{L}}$ say. Each supporting d-surface will be specified by a mapping $\sigma \mapsto x\{\sigma\}$ giving the local background coordinates x^μ (μ=0, ... , n–1) as functions of local internal coordinates σ^i (i=0, ... , d–1). The corresponding d-dimensional surface metric tensor ${}^{(\rm d)}\gamma_{ij}$ that is induced (in the manner described in Subsection 1.1) as the pull back of the n-dimensional background spacetime metric $g_{\mu\nu}$, will determine the natural surface measure, ${}^{(\rm d)}d\overline{\mathcal{S}}$, in terms of which the total action will be expressible in the form

$$\mathcal{I} = \sum_{\rm d} \int {}^{(\rm d)}d\overline{\mathcal{S}}\ {}^{(\rm d)}\overline{\mathcal{L}}\ , \qquad {}^{(\rm d)}d\overline{\mathcal{S}} = \sqrt{\Vert {}^{(\rm d)}\gamma \Vert}\, d^{\rm d}\sigma\, . \tag{83}$$

As a formal artifice whose use is an unnecessary complication in ordinary dynamical calculations but that can be useful for purposes such as the calculation of radiation, the *confined* (d-surface supported) but locally *regular* Lagrangian scalar fields ${}^{(\mathrm{d})}\overline{\mathcal{L}}$ can be replaced by corresponding unconfined, so no longer regular but *distributional* fields ${}^{(\mathrm{d})}\hat{\mathcal{L}}$, in order to allow the the basic multidimensional action (83) to be represented as a single integral,

$$\mathcal{I} = \int d\mathcal{S} \sum_{\mathrm{d}} {}^{(\mathrm{d})}\hat{\mathcal{L}} \, , \qquad d\mathcal{S} = \sqrt{\|g\|}\, d^{\mathrm{n}}x \, . \tag{84}$$

over the n-dimensional *background* spacetime. In order to do this, it is evident that for each (d–1)-brane of the complex the required distributional action contribution ${}^{(\mathrm{d})}\hat{\mathcal{L}}$ must be constructed in terms of the corresponding regular d-surface density scalar ${}^{(\mathrm{d})}\overline{\mathcal{L}}$ according to the prescription that is expressible in standard Dirac notation as

$${}^{(\mathrm{d})}\hat{\mathcal{L}} = \|g\|^{-1/2} \int {}^{(\mathrm{d})}d\overline{\mathcal{S}}\ {}^{(\mathrm{d})}\overline{\mathcal{L}}\, \delta^{\mathrm{n}}[x - x\{\sigma\}] \, . \tag{85}$$

2.2 Current, vorticity, and stress-energy tensor

In the kind of model under consideration, each supporting d-surface is supposed to be endowed with its own independent internal field variables which are allowed to couple with each other and with their derivatives in the corresponding d-surface Lagrangian contribution ${}^{(\mathrm{d})}\overline{\mathcal{L}}$, and which are also allowed to couple into the Lagrangian contribution ${}^{(\mathrm{d-1})}\overline{\mathcal{L}}$ on any of its attached boundary (d–1) surfaces, though – in order not to violate the strong dynamic regularity condition – they are not allowed to couple into contributions of dimension (d–2) or lower. As well as involving its own d-brane surface fields and those of any (d+1) brane to whose boundary it may belong, each contribution ${}^{(\mathrm{d})}\overline{\mathcal{L}}$ may also depend passively on the fields of a fixed higher dimensional background. Such fields will of course always include the background spacetime metric $g_{\mu\nu}$ itself. Apart from that, the most commonly relevant kind of backround field (the only one allowed for in the earlier analysis, [5]) is a Maxwellian gauge potential A_μ whose exterior derivative is the automatically "closed" electromagnetic field,

$$F_{\mu\nu} = 2\nabla_{[\mu} A_{\nu]} \, , \qquad \nabla_{[\mu} F_{\nu\rho]} = 0 \, . \tag{86}$$

Although many other possibilities can in principle be enviseaged, for the sake of simplicity the following analysis will not go beyond allowance for the only one that is important in applications to the kind of cosmic or superfluid defects that are the subject of discussion in the present volume, namely an antisymmetric Kalb-Ramond gauge field $B_{\mu\nu} = -B_{\nu\mu}$ whose exterior derivative is an automativally closed physical current 3-form,

$$N_{\mu\nu\rho} = 3\nabla_{[\mu} B_{\nu\rho]} \, , \qquad \nabla_{[\mu} N_{\nu\rho\sigma]} = 0 \, . \tag{87}$$

Just as a Maxwellian gauge transformation of the form $A_\mu \mapsto A_\mu + \nabla_\mu \alpha$ for an arbitrary scalar α leaves the electromagnetic field (86) invariant, so analogously

a Kalb-Ramond gauge transformation $B_{\mu\nu} \mapsto B_{\mu\nu} + 2\nabla_{[\mu}\chi_{\nu]}$ for an arbitrary covector χ_μ leaves the corresponding current 3-form (87) invariant. In applications to ordinary 4-dimensional spacetime, the current 3-form will just be the dual $N_{\mu\nu\rho} = \varepsilon_{\mu\nu\rho\sigma}N^\sigma$ of an ordinary current vector N^μ satisfying a conservation law of the usual type, $\nabla_\mu N^\mu = 0$. Such a Kalb-Ramond representation can be used to provide an elegant variational formulation for ordinary perfect fluid theory [31] and is particularly convenient for setting up "global" string models of vortices both in a simple cosmic axion or Higgs field [32], [33], [34] and in a superfluid [35] such as liquid Helium-4.

In accordance with the preceeding considerations, the analysis that follows will be based on the postulate that the action is covariantly and gauge invariantly determined by specifying each scalar Lagrangian contribution ${}^{(\mathrm{d})}\overline{\mathcal{L}}$ as a function just of the background fields, A_μ, $B_{\mu\nu}$ and of course $g_{\mu\nu}$, and of any relevant internal fields (which in the simplest non-trivial case – exemplified by string models [36], [37] of the category needed for the macroscopic description of Witten type [8] superconducting vortices – consist just of a phase scalar φ). In accordance with the restriction that the branes be "pure" or "fine" in the sense explained above, it is postulated that polarisation effects are excluded by ruling out couplings involving gradients of the background fields. This means that the effect of making arbitrary infinitesimal "Lagrangian" variations $\underset{\mathrm{L}}{\delta}A_\mu$, $\underset{\mathrm{L}}{\delta}B_{\mu\nu}$, $\underset{\mathrm{L}}{\delta}g_{\mu\nu}$ of the background fields will be to induce a corresponding variation $\delta\mathcal{I}$ of the action that simply has the form

$$\delta\mathcal{I} = \sum_{\mathrm{d}} \int {}^{(\mathrm{d})}\!d\overline{\mathcal{S}} \left\{ {}^{(\mathrm{d})}\overline{j}^{\mu}\underset{\mathrm{L}}{\delta}A_\mu + \tfrac{1}{2}\, {}^{(\mathrm{d})}\overline{w}^{\mu\nu}\underset{\mathrm{L}}{\delta}B_{\mu\nu} + \tfrac{1}{2}\, {}^{(\mathrm{d})}\overline{T}^{\mu\nu}\underset{\mathrm{L}}{\delta}g_{\mu\nu} \right\} , \tag{88}$$

provided either that that the relevant independent internal field components are fixed or else that the internal dynamic equations of motion are satisfied in accordance with the variational principle stipulating that variations of the relevant independent field variables should make no difference. For each d-brane of the complex, this partial differentiation formula implicitly specifies the corresponding *electromagnetic surface current density* vector ${}^{(\mathrm{d})}\overline{j}^{\mu}$, the *surface vorticity flux* bivector ${}^{(\mathrm{d})}\overline{w}^{\mu\nu} = -\,{}^{(\mathrm{d})}\overline{w}^{\nu\mu}$, and the *surface stress momentum energy density* tensor ${}^{(\mathrm{d})}\overline{T}^{\mu\nu} = {}^{(\mathrm{d})}\overline{T}^{\nu\mu}$, which are formally expressible more explicitly as

$${}^{(\mathrm{d})}\overline{j}^{\mu} = \frac{\partial\, {}^{(\mathrm{d})}\overline{\mathcal{L}}}{\partial A_\mu} , \qquad {}^{(\mathrm{d})}\overline{w}^{\mu\nu} = 2\frac{\partial\, {}^{(\mathrm{d})}\overline{\mathcal{L}}}{\partial B_{\mu\nu}} , \tag{89}$$

and

$${}^{(\mathrm{d})}\overline{T}^{\mu\nu} = 2\frac{\partial\, {}^{(\mathrm{d})}\overline{\mathcal{L}}}{\partial g_{\mu\nu}} + {}^{(\mathrm{d})}\overline{\mathcal{L}}\, {}^{(\mathrm{d})}\eta^{\mu\nu} , \tag{90}$$

of which the latter is obtained using the formula

$$\underset{\mathrm{L}}{\delta}\big({}^{(\mathrm{d})}\!d\overline{\mathcal{S}}\big) = \tfrac{1}{2}\, {}^{(\mathrm{d})}\eta^{\mu\nu}\big(\underset{\mathrm{L}}{\delta}g_{\mu\nu}\big)\, {}^{(\mathrm{d})}\!d\overline{\mathcal{S}} , \tag{91}$$

where ${}^{(\mathrm{d})}\eta^{\mu\nu}$ is the rank-d *fundamental tensor* of the d-dimensional imbedding, as defined in the manner described in the preceeding section.

2.3 Conservation of current and vorticity

The condition that the action be gauge invariant means that if one simply sets $\underset{\mathrm{L}}{\delta} A_\mu = \nabla_\mu \alpha$, $\underset{\mathrm{L}}{\delta} B_{\mu\nu} = 2\nabla_{[\mu}\chi_{\nu]}$, $d_{\mathrm{L}} g_{\mu\nu} = 0$, for arbitrarily chosen α and χ_μ then $\delta\mathcal{I}$ should simply vanish, i.e.

$$\sum_{\mathrm{d}} \int d^{(\mathrm{d})}\overline{\mathcal{S}} \left\{ {}^{(\mathrm{d})}\overline{j}^{\mu} \nabla_\mu \alpha + {}^{(\mathrm{d})}\overline{w}^{\mu\nu} \nabla_\mu \chi_\nu \right\} = 0\,. \tag{92}$$

In order for this to be able to hold for all possible fields α and χ_μ it is evident that the surface current ${}^{(\mathrm{d})}\overline{j}^{\mu}$ and the vorticity flux bivector ${}^{(\mathrm{d})}\overline{w}^{\mu\nu}$ must (as one would anyway expect from the consideration that they depend just on the relevant internal d-surface fields) be purely d-surface tangential, i.e. their contractions with the relevant rank (n–d) orthogonal projector ${}^{(\mathrm{d})}\!\perp^{\mu}{}_{\nu} = g^{\mu}{}_{\nu} - {}^{(\mathrm{d})}\eta^{\mu}{}_{\nu}$ must vanish:

$$ {}^{(\mathrm{d})}\!\perp^{\mu}{}_{\nu}\, {}^{(\mathrm{d})}\overline{j}^{\nu} = 0\,, \qquad {}^{(\mathrm{d})}\!\perp^{\mu}{}_{\nu}\, {}^{(\mathrm{d})}\overline{w}^{\nu\rho} = 0\,. \tag{93}$$

Hence, decomposing the full gradient operator ∇_μ as the sum of its tangentially projected part ${}^{(\mathrm{d})}\overline{\nabla}_\mu = {}^{(\mathrm{d})}\eta^{\nu}{}_{\mu}\nabla_\nu$ and of its orthogonally projected part ${}^{(\mathrm{d})}\!\perp^{\nu}{}_{\mu}\nabla_\mu$, and noting that by (93) the latter will give no contribution, one sees that (92) will take the form

$$\sum_{\mathrm{d}} \int {}^{(\mathrm{d})}d\overline{\mathcal{S}} \left\{ {}^{(\mathrm{d})}\overline{\nabla}_\mu \left({}^{(\mathrm{d})}\overline{j}^{\mu}\alpha + {}^{(\mathrm{d})}\overline{w}^{\mu\nu}\chi_\nu \right) - \alpha\, {}^{(\mathrm{d})}\overline{\nabla}_\mu \overline{j}^{\mu} - \chi_\nu\, {}^{(\mathrm{d})}\overline{\nabla}_\mu\, {}^{(\mathrm{d})}\overline{w}^{\mu\nu} \right\} = 0\,, \tag{94}$$

in which first term of each integrand is a pure surface divergence. Such a divergence can be dealt with using Green's theorem, according to which, for any d-dimensional support surface ${}^{(\mathrm{d})}\overline{\mathcal{S}}$ of a (d–1)-brane, one has the identity

$$\int {}^{(\mathrm{d})}d\overline{\mathcal{S}}\, {}^{(\mathrm{d})}\overline{\nabla}_\mu\, {}^{(\mathrm{d})}\overline{j}^{\mu} = \oint {}^{(\mathrm{d}-1)}d\overline{\mathcal{S}}\, {}^{(\mathrm{d})}\lambda_\mu\, {}^{(\mathrm{d})}\overline{j}^{\mu}\,, \tag{95}$$

where the integral on the right is taken over the boundary (d–1)-surface of $\partial\,{}^{(\mathrm{d})}\overline{\mathcal{S}}$ of ${}^{(\mathrm{d})}\overline{\mathcal{S}}$, and ${}^{(\mathrm{d})}\lambda_\mu$ is the (uniquely defined) unit tangent vector on the d-surface that is directed normally outwards at its (d–1)-dimensional boundary. Bearing in mind that a membrane support hypersurface can belong to the boundary of two distinct media, and that for d$\leq$ n–3 a d-brane may belong to a common boundary joining three or more distinct (d+1)-branes of the complex under consideration, one sees that (94) is equivalent to the condition

$$\begin{aligned}\sum_{\mathrm{p}} \int {}^{(\mathrm{p})}d\overline{\mathcal{S}} \Big\{ \alpha \Big({}^{(\mathrm{p})}\overline{\nabla}_\mu\, {}^{(\mathrm{p})}\overline{j}^{\mu} - \sum_{\mathrm{d=p+1}} {}^{(\mathrm{d})}\lambda_\mu\, {}^{(\mathrm{d})}\overline{j}^{\mu} \Big) & \\ + \chi_\nu \Big({}^{(\mathrm{p})}\overline{\nabla}_\mu\, {}^{(\mathrm{p})}\overline{w}^{\mu\nu} - \sum_{\mathrm{d=p+1}} {}^{(\mathrm{d})}\lambda_\mu\, {}^{(\mathrm{d})}\overline{w}^{\mu\nu} \Big) \Big\} &= 0\,, \end{aligned} \tag{96}$$

where, for a particular p-dimensionally supported (p–1)-brane, the summation "over d=p+1" is to be understood as consisting of a contribution from each

(p+1)-dimensionally supported p-brane attached to it, where for each such p-brane, ${}^{(\mathrm{d})}\lambda_\mu$ denotes the (uniquely defined) unit tangent vector on its (p+1)-dimensional support surface that is directed normally towards the p-dimensional support surface of the boundary (p–1)-brane. The Maxwell gauge invariance requirement to the effect that (96) should hold for arbitrary α can be seen to entail an electromagnetic charge conservation law of the form

$$ {}^{(\mathrm{p})}\overline{\nabla}_\mu\, {}^{(\mathrm{p})}\overline{j}^\mu = \sum_{\mathrm{d=p+1}} {}^{(\mathrm{d})}\lambda_\mu\, {}^{(\mathrm{d})}\overline{j}^\mu \,. \tag{97} $$

This can be seen from (95) to be be interpretable as meaning that the total charge flowing out of particular (d–1)-brane from its boundary is balanced by the total charge flowing into it from any d-branes to which it may be attached. The analogous Kalb-Ramond gauge invariance requirement that (96) should also hold for arbitrary χ_μ can be seen to entail a corresponding vorticity conservation law of the form

$$ {}^{(\mathrm{p})}\overline{\nabla}_\mu\, {}^{(\mathrm{p})}\overline{w}^{\mu\nu} = \sum_{\mathrm{d=p+1}} {}^{(\mathrm{d})}\lambda_\mu\, {}^{(\mathrm{d})}\overline{w}^{\mu\nu} \,. \tag{98} $$

A more sophisticated but less practical way of deriving the foregoing conservation laws would be to work not from the expression (83) in terms of ordinary surface integrals but instead to use the superficially simpler expression (84) in terms of distributions, which leads to the replacement of (97) by the ultimately equivalent (more formally obvious but less directly meaningful) expression

$$ \nabla_\mu \Big(\sum_{\mathrm{d}} {}^{(\mathrm{d})}\hat{j}^\mu \Big) = 0 \tag{99} $$

involving the no longer regular but Dirac distributional current ${}^{(\mathrm{d})}\hat{j}^\mu$ that is given in terms of the corresponding regular surface current ${}^{(\mathrm{d})}\overline{j}^\mu$ by

$$ {}^{(\mathrm{d})}\hat{j}^\mu = \Vert g \Vert^{-1/2} \int {}^{(\mathrm{d})}\!d\overline{\mathcal{S}}\ {}^{(\mathrm{d})}\overline{j}^\mu \delta^{\mathrm{n}}[x - x\{\sigma\}] \,. \tag{100} $$

Similarly one can if one wishes rewrite the vorticity flux conservation law (98) in the distributional form

$$ \nabla_\mu \Big(\sum {}^{(\mathrm{d})}\hat{w}^{\mu\nu} \Big) = 0 \,, \tag{101} $$

where the distributional vorticity flux ${}^{(\mathrm{d})}\hat{w}^{\mu\nu}$ is given in terms of the corresponding regular surface flux ${}^{(\mathrm{d})}\overline{w}^{\mu\nu}$ by

$$ {}^{(\mathrm{d})}\hat{w}^{\mu\nu} = \Vert g \Vert^{-1/2} \int {}^{(\mathrm{d})}\!d\overline{\mathcal{S}}\ {}^{(\mathrm{d})}\overline{w}^{\mu\nu} \delta^{\mathrm{n}}[x - x\{\sigma\}] \,. \tag{102} $$

It is left as an entirely optional exercise for any readers who may be adept in distribution theory to show how the ordinary functional relationships (97) and (98) can be recovered by by integrating out the Dirac distributions in (99) and (101).

2.4 Force and the stress balance equation

The condition that the hypothetical variations introduced in (88) should be "Lagrangian" simply means that they are to be understood to be measured with respect to a reference system that is comoving with the various branes under consideration, so that their localisation with respect to it remains fixed. This condition is necessary for the variation to be meaningly definable at all for a field whose support is confined to a particular brane locus, but in the case of an unrestricted background field one can enviseage the alternative possibility of an "Eulerian" variation, meaning one defined with respect to a reference system that is fixed in advance, independently of the localisation of the brane complex, the standard example being that of a Minkowski reference system in the case of a background that is flat. In such a case the relation between the more generally meaningfull Lagrangian (comoving) variation, denoted by $\underset{\rm L}{\delta}$, and the corresponding Eulerian (fixed point) variation denoted by $\underset{\rm E}{\delta}$ say will be given by Lie differentiation with respect to the vector field ξ^μ say that specifies the infinitesimal of the comoving reference system with respect to the fixed background, i.e. one has

$$\underset{\rm L}{\delta} - \underset{\rm E}{\delta} = \vec{\xi}\!\pounds\,, \tag{103}$$

where the Lie differentiation operator $\vec{\xi}\!\pounds$ is given for the background fields under consideration here by

$$\vec{\xi}\!\pounds A_\mu = \xi^\nu \nabla_\nu A_\mu + A_\nu \nabla_\mu \xi^\nu\,, \tag{104}$$

$$\vec{\xi}\!\pounds B_{\mu\nu} = \xi^\nu \nabla_\nu B_{\mu\nu} + 2 B_{\rho[\nu} \nabla_{\mu]} \xi^\rho\,, \tag{105}$$

$$\vec{\xi}\!\pounds g_{\mu\nu} = 2\nabla_{(\mu}\xi_{\nu)}\,. \tag{106}$$

This brings us to the main point of this section which is the derivation of the dynamic equations governing the extrinsic motion of the branes of the complex, which are obtained from the variational principle to the effect that the action $\mathcal{I}$ is left invariant not only by infinitesimal variations of the relevant independent intrinsic fields on the support surfaces but also by infinitesimal displacements of the support surfaces themselves. Since the background fields A_μ, $B_{\mu\nu}$, and $g_{\mu\nu}$ are to be considered as fixed, the relevant Eulerian variations simply vanish, and so the resulting Lagrangian variations will be directly identifiable with the corresponding Lie derivatives – as given by (106) – with respect to the generating vector field ξ^μ of the infinitesimal displacement under consideration. The variational principle governing the equations of extrinsic motion is thus obtained by setting to zero the result of substituting these Lie derivatives in place of the corresponding Lagrangian variations in the more general variation formula (88), which gives

$$\sum_{\rm d} \int {}^{(\rm d)}\!d\overline{\mathcal{S}} \left\{ {}^{(\rm d)}\overline{j}^{\mu}\ \vec{\xi}\!\pounds A_\mu + \tfrac{1}{2}\, {}^{(\rm d)}\overline{w}^{\mu\nu}\ \vec{\xi}\!\pounds B_{\mu\nu} + \tfrac{1}{2}\, {}^{(\rm d)}\overline{T}^{\mu\nu}\ \vec{\xi}\!\pounds g_{\mu\nu} \right\} = 0\,. \tag{107}$$

The requirement that this should hold for any choice of ξ^μ evidently implies that the tangentiality conditions (93) for the surface fluxes ${}^{(\rm d)}\overline{j}^{\mu}$ and ${}^{(\rm d)}\overline{w}^{\mu\nu}$ must be

supplemented by an analogous d-surface tangentiality condition for the surface stress momentum energy tensor ${}^{(\mathrm{d})}\overline{T}^{\mu\nu}$, which must satisfy

$$ {}^{(\mathrm{d})}\!\perp^{\mu}_{\ \nu}\, {}^{(\mathrm{d})}\overline{T}^{\nu\rho} = 0\,. \tag{108} $$

(as again one would expect anyway from the consideration that it depends just on the relevant internal d-surface fields). This allows (106) to be written out in the form

$$ \sum_{\mathrm{d}} \int {}^{(\mathrm{d})}\!d\overline{\mathcal{S}} \Big\{ \xi^\rho \Big(F_{\rho\mu}\, {}^{(\mathrm{d})}\overline{j}^{\mu} + \tfrac{1}{2} N_{\rho\mu\nu}\, {}^{(\mathrm{d})}\overline{w}^{\mu\nu} \\ - {}^{(\mathrm{d})}\overline{\nabla}_{\mu}\, {}^{(\mathrm{d})}\overline{T}^{\mu}_{\rho} - A_\rho\, {}^{(\mathrm{d})}\overline{\nabla}_{\mu}\, {}^{(\mathrm{d})}\overline{j}^{\mu} - B_{\rho\nu}\, {}^{(\mathrm{d})}\overline{\nabla}_{\mu}\, {}^{(\mathrm{d})}\overline{w}^{\mu\nu} \Big) \\ + {}^{(\mathrm{d})}\overline{\nabla}_{\mu} \Big(\xi^\rho \big(A_\rho\, {}^{(\mathrm{d})}\overline{j}^{\mu} + B_{\rho\nu}\, {}^{(\mathrm{d})}\overline{w}^{\mu\nu} + {}^{(\mathrm{d})}\overline{T}^{\mu}{}_{\rho} \big) \Big) \Big\} \quad = \quad 0\,, \tag{109} $$

in which the final contribution is a pure surface divergence that can be dealt with using Green's theorem as before. Using the results (97) and (98) of the analysis of the consequences of gauge invariance and proceeding as in their derivation above, one sees that the condition for (109) to hold for an arbitrary field ξ^μ is that, on each (p–1)-brane of the complex, the dynamical equations

$$ {}^{(\mathrm{p})}\overline{\nabla}_{\mu}\, {}^{(\mathrm{p})}\overline{T}^{\mu}{}_{\rho} = {}^{(\mathrm{p})}f_{\rho}\,, \tag{110} $$

should be satisfied for a total force density ${}^{(\mathrm{p})}f_{\rho}$ given by

$$ {}^{(\mathrm{p})}f_{\rho} = {}^{(\mathrm{p})}\overline{f}_{\rho} + {}^{(\mathrm{p})}\check{f}_{\rho}\,, \tag{111} $$

where ${}^{(\mathrm{p})}\check{f}_{\rho}$ is the contribution of the contact force exerted on the p-surface by other members of the brane complex, which takes the form

$$ {}^{(\mathrm{p})}\check{f}_{\rho} = \sum_{\mathrm{d=p+1}} {}^{(\mathrm{d})}\lambda_{\mu}\, {}^{(\mathrm{d})}\overline{T}^{\mu}{}_{\rho}\,, \tag{112} $$

while the other force density contribution ${}^{(\mathrm{p})}\overline{f}_{\rho}$ represents the effect of the external background fields, which is given by

$$ {}^{(\mathrm{p})}\overline{f}_{\rho} = F_{\rho\mu}\, {}^{(\mathrm{p})}\overline{j}^{\mu} + \tfrac{1}{2} N_{\rho\mu\nu}\, {}^{(\mathrm{p})}\overline{w}^{\mu\nu}\,. \tag{113} $$

As before, the summation "over d=p+1" in (112) is to be understood as consisting of a contribution from each of the p-branes attached to the (p–1)- brane under consideration, where for each such attached p-brane, ${}^{(\mathrm{d})}\lambda_{\mu}$ denotes the (uniquely defined) unit tangent vector on its (p+1)-dimensional support surface that is directed normally towards the p-dimensional support surface of the boundary (p–1)-brane. The first of the background force contibutions in (113) is of course the Lorentz type force density resulting from the effect of the electromagnetic field on the surface current, while the other contribution in (113) is a Joukowsky type force density (of the kind responsible for the lift on an aerofoil)

resulting from the Magnus effect, which acts in the case of a "global" string [32], [33] though not in the case of a string of the "local" type for which the relevant vorticity flux ${}^{(\rm p)}\overline{w}^{\mu\nu}$ will be zero. As with the conservation laws (97) and (98), so also the explicit force density balance law expressed by (110) can alternatively be expressed in terms of the corresponding Dirac distributional stress momentum energy and background force density tensors, ${}^{(\rm d)}\hat T^{\mu\nu}$ and ${}^{(\rm d)}\hat f_\mu$, which are given for each (d–1)-brane in terms of the corresponding regular surface stress momentum energy and background force density tensors ${}^{(\rm d)}\overline{T}^{\mu\nu}$ and ${}^{(\rm d)}\overline{f}_\mu$ by

$$ {}^{(\rm d)}\hat T^{\mu\nu} = \Vert g\Vert^{-1/2}\int {}^{(\rm d)}\!d\overline{\mathcal S}\ {}^{(\rm d)}\overline{T}^{\mu\nu}\delta^{\rm n}[x-x\{\sigma\}] \tag{114}$$

and

$$ {}^{(\rm d)}\hat f_\mu = \Vert g\Vert^{-1/2}\int {}^{(\rm d)}\!d\overline{\mathcal S}\ {}^{(\rm d)}\overline{f}_\mu\delta^{\rm n}[x-x\{\sigma\}]\,. \tag{115}$$

The equivalent – more formally obvious but less explicitly meaningful – distributional versional version of the force balance law (110) takes the form

$$ \nabla_\mu\Big(\sum_{\rm d}{}^{(\rm d)}\hat T^{\mu\nu}\Big) = \hat f_\mu\,, \tag{116}$$

where the total Dirac distributional force density is given in terms of the electromagnetic current distributions (100) and the vorticity flux distributions (102) by

$$ \hat f_\mu = F_{\rho\mu}\sum_{\rm d}{}^{(\rm d)}\hat\jmath^\mu + \tfrac{1}{2}N_{\rho\mu\nu}\sum_{\rm d}{}^{(\rm d)}\hat w^{\mu\nu}\,. \tag{117}$$

It is again left as an optional exercise for readers who are adept in the use of Dirac distributions to show that the system (110), (112), (113) is obtainable from (116) and (117) by substituting (100), (102), (114), (115).

As an immediate corollary of (110), it is to be noted that for any vector field ℓ^μ that generates a continuous symmetry of the background spacetime metric, i.e. for any solution of the Killing equations

$$ \nabla_{(\mu}\ell_{\nu)} = 0\,, \tag{118}$$

one can construct a corresponding surface momentum or energy density current

$$ {}^{(\rm p)}\overline{P}^\mu = {}^{(\rm p)}\overline{T}^{\mu\nu}\ell_\nu\ , \tag{119}$$

that will satisfy

$$ {}^{(\rm p)}\overline{\nabla}_\mu\,{}^{(\rm p)}\overline{P}^\mu = \sum_{\rm d=p+1}{}^{(\rm d)}\lambda_\mu\,{}^{(\rm d)}\overline{P}^\mu + {}^{(\rm p)}\overline{f}_\mu k^\mu\,. \tag{120}$$

In typical applications for which the n-dimensional background spacetime can be taken to be flat there will be n independent translation Killing vectors which alone (without recourse to the further n(n–1)/2 rotation and boost Killing vectors of the Lorentz algebra) will provide a set of relations of the form (120) that together provide the same information as that in the full force balance equation (110) or (116).

2.5 The equation of extrinsic motion

Rather than the distributional version (116), it is the explicit version (110) of the force balance law that is directly useful for calculating the dynamic evolution of the brane support surfaces. Since the relation (116) involves n independent components whereas the support surface involved is only p-dimensional, there is a certain redundancy, which results from the fact that if the virtual displacement field ξ^μ is tangential to the surface in question it cannot affect the action. Thus if ${}^{(p)}\!\perp^{\mu}_{\ \nu}\xi^\nu = 0$, the condition (107) will be satisfied as a mere identity – provided of course that the field equations governing the internal fields of the system are satisfied. It follows that the non-redundent information governing the extrinsic motion of the p-dimensional support surface will be given just by the orthogonally projected part of (110). Integrating by parts, using the fact that, by (6) and (17), the surface gradient of the rank-(n–p) orthogonal projector ${}^{(p)}\!\perp^{\mu}_{\ \nu}$ will be given in terms of the second fundamental tensor ${}^{(p)}\!K_{\mu\nu}{}^{\rho}$ of the p-surface by

$$ {}^{(p)}\overline{\nabla}_{\mu}\, {}^{(p)}\!\perp^{\nu}_{\ \rho} = -\, {}^{(p)}\!K_{\mu\nu}{}^{\rho} - \, {}^{(p)}\!K_{\mu}{}^{\rho}{}_{\nu}\,, \tag{121} $$

it can be seen that the extrinsic equations of motion obtained as the orthogonally projected part of (110) will finally be expressible by

$$ {}^{(p)}\overline{T}^{\mu\nu}\, {}^{(p)}\!K_{\mu\nu}{}^{\rho} = {}^{(p)}\!\perp^{\rho}_{\ \mu}\, {}^{(p)}\!f^{\mu}\,. \tag{122} $$

It is to be emphasised that the formal validity of the formula that has just been derived is not confined to the variational models on which the above derivation is based, but also extends to dissipative models (involving effects such as external drag by the background medium [40], [38], [39] or mutual resistance between independent internal currents). The condition that even a non-conservative macroscopic model should be compatible with an underlying microscopic model of conservative type requires the existence (representing to averages of corresponding microscopic quantities) of appropriate stress momentum energy density and force density fields satisfying (122).

The ubiquitously applicable formula (122)is interpretable as being just the natural higher generalisation of "Newton's law" (equating the product of mass with acceleration to the applied force) in the case of a particle. The surface stress momentum energy tensor, ${}^{(p)}\overline{T}^{\mu\nu}$, generalises the mass, and the second fundamental tensor, ${}^{(p)}\!K_{\mu\nu}{}^{\rho}$, generalises the acceleration.

The way this works out in the 1-dimensional case of a "pure" point particle (i.e. a monopole) of mass m, for which the Lagrangian is given simply by ${}^{(1)}\overline{\mathcal{L}} = -m$, is as follows. The 1-dimensional energy tensor will be obtained in terms of the unit tangent vector u^μ ($u^\mu u_\mu = -1$) as ${}^{(1)}\overline{T}^{\mu\nu} = m\, u^\mu u^\nu$, and in this zero-brane case, the first fundamental tensor will simply be given by ${}^{(1)}\eta^{\mu\nu} = -u^\mu u^\nu$, so that the second fundamental tensor will be obtained in terms of the acceleration $\dot{u}^\mu = u^\nu \nabla_\nu u^\mu$ as ${}^{(1)}\!K_{\mu\nu}{}^{\rho} = u_\mu u_\nu \dot{u}^\rho$. Thus (122) can be seen to reduce in the case of a particle simply to the usual familiar form $m\,\dot{u}^\rho ={}^{(1)}\!\perp^{\rho}_{\ \mu}\,{}^{(1)}\!f^{\mu}$.

3 Perturbations and curvature effects beyond the pure brane limit

3.1 First order perturbations and the extrinsic characteristic equation

Two of the most useful formulae for the analysis of small perturbations of a string or higher brane worldsheet are the expressions for the infinitesimal Lagrangian (comoving) variation of the first and second fundamental tensors in terms of the corresponding comoving variation $\underset{\mathrm{L}}{\delta} g_{\mu\nu}$ of the metric (with respect to the comoving reference system). For the first fundamental tensor one easily obtains

$$\underset{\mathrm{L}}{\delta}\eta^{\mu\nu} = -\eta^{\mu\rho}\eta^{\mu\sigma}\underset{\mathrm{L}}{\delta} g_{\rho\sigma} \; , \qquad \underset{\mathrm{L}}{\delta}\eta^{\mu}{}_{\nu} = \eta^{\mu\rho}\perp^{\sigma}_{\nu}\underset{\mathrm{L}}{\delta} g_{\rho\sigma} \tag{123}$$

and, by substituting this in the defining relation (14), the corresponding Lagrangian variation of the second fundamental tensor is obtained [44] as

$$\underset{\mathrm{L}}{\delta} K_{\mu\nu}{}^{\rho} = \perp^{\rho}_{\lambda}\eta^{\sigma}_{\mu}\eta^{\tau}_{\nu}\underset{\mathrm{L}}{\delta}\Gamma_{\sigma}{}^{\lambda}{}_{\tau} + \big(2\perp^{\sigma}_{(\mu} K_{\nu)}{}^{\tau\rho} - K_{\mu\nu}{}^{\sigma}\eta^{\tau\rho}\big)\underset{\mathrm{L}}{\delta} g_{\sigma\tau} \; , \tag{124}$$

where the Lagrangian variation of the connection (20) is given by the well known formula

$$\underset{\mathrm{L}}{\delta}\Gamma_{\sigma}{}^{\lambda}{}_{\tau} = g^{\lambda\rho}\big(\nabla_{(\sigma}\underset{\mathrm{L}}{\delta} g_{\tau)\rho} - \tfrac{1}{2}\nabla_{\rho}\underset{\mathrm{L}}{\delta} g_{\sigma\tau}\big) \, . \tag{125}$$

Since we are concerned here only with cases for which the background is fixed in advance so that the Eulerian variation d_{E} will vanish in (103), the Lagrangian variation of the metric will be given just by its Lie derivative with respect to the infinitesimal displacement vector field ξ^{μ} that generates the displacement of the worldsheet under consideration, i.e. we shall simply have

$$\underset{\mathrm{L}}{\delta} g_{\sigma\tau} = 2\nabla_{(\sigma}\xi_{\tau)} \, . \tag{126}$$

It then follows from (125) that the Lagrangian variation of the connection will be given by

$$\underset{\mathrm{L}}{\delta}\Gamma_{\sigma}{}^{\lambda}{}_{\tau} = \nabla_{(\sigma}\nabla_{\tau)}\xi^{\lambda} - \mathcal{R}^{\lambda}{}_{(\sigma\tau)\rho}\xi^{\rho} \; , \tag{127}$$

where $\mathcal{R}^{\lambda}{}_{\sigma\tau\rho}$ is the background Riemann curvature (which will be negligible in typical applications for which the lengthscales characterising the geometric features of interest will be small compared with those characterising any background spacetime curvature). The Lagrangian variation of the first fundamental tensor is thus finally obtained in the form

$$\underset{\mathrm{L}}{\delta}\eta^{\mu\nu} = -2\eta^{(\mu}_{\sigma}\overline{\nabla}^{\nu)}\xi^{\sigma} \; , \tag{128}$$

while that of the second fundamental tensor is found to be given by

$$\underset{\mathrm{L}}{\delta} K_{\mu\nu}{}^{\rho} = \perp^{\rho}_{\lambda}\big(\overline{\nabla}_{(\mu}\overline{\nabla}_{\nu)}\xi^{\lambda} - \eta^{\sigma}_{(\mu}\eta^{\tau}{}_{\nu)}\mathcal{R}^{\lambda}{}_{\sigma\tau\rho}\xi^{\rho} - K^{\sigma}{}_{(\mu\nu)}\overline{\nabla}_{\sigma}\xi^{\lambda}\big) + \tag{129}$$

$$\big(2\perp^{\sigma}_{(\mu} K_{\nu)\tau}{}^{\rho} - g^{\rho}_{\tau} K_{\mu\nu}{}^{\sigma}\big)\big(\nabla_{\sigma}\xi^{\tau} + \overline{\nabla}^{\tau}\xi_{\sigma}\big) \, . \tag{130}$$

It is instructive to apply the forgoing formulae to the case of a *free* pure brane worldsheet, meaning one for which there is no external force contribution so that the equation of extinsic motion reduces to the form

$$\overline{T}^{\mu\nu} K_{\mu\nu}{}^{\rho} = 0\,. \tag{131}$$

On varying the relation (131) using (130) in conjunction with the orthogonality property (108) and the unperturbed equation (131) itself, the equation governing the propagation of the infinitesimal displacement vector is obtained in the form

$$\perp^{\!\rho}_{\lambda}\overline{T}^{\mu\nu}\big(\overline{\nabla}_{\mu}\overline{\nabla}_{\nu}\xi^{\lambda} - \mathcal{R}^{\lambda}{}_{\mu\nu\sigma}\xi^{\sigma}\big) = -K_{\mu\nu}{}^{\rho}\,\underset{\rm L}{\delta}\overline{T}^{\mu\nu}\ . \tag{132}$$

The extrinsic perturbation equation (132) is by itself only part of the complete system of perturbation equations governing the evolution of the brane, the remaining equations of the system being those governing the evolution of whatever surface current [41] and other relevant internal fields on the supporting worldsheet may be relevant. The perturbations of such fields are involved in the source term on the right of (132), whose explicit evaluation depends on the specific form of the relevant currents or other internal fields. However it is not necessary to know the specific form of such internal fields for the purpose just of deriving the characteristic velocities of propagation of the extrinsic propagations represented by the displacement vector ξ^{μ}, so long as they contribute to the source term on the right of the linearised perturbation equation (132) only at first differential order, so that the characteristic velocities will be completely determined by the first term on the left of (132) which will be the only second differential order contribution. It is apparent from (132) that under these conditions the equation for the characteristic tangent covector χ_{μ} say will be given independently of any details of the surface currents or other internal fields simply [5] by

$$\overline{T}^{\mu\nu}\chi_{\mu}\chi_{\nu} = 0\,. \tag{133}$$

(It can be seen that the unperturbed surface stress momentum energy density tensor $\overline{T}^{\mu\nu}$ plays the same role here as that of the unperturbed metric tensor $g^{\mu\nu}$ in the analogous characteristic equation for the familiar case of a massless background spacetime field, as exemplified by electromagnetic or gravitational radiation.)

3.2 *Higher order displacements by "straight" transportation

Mere linear perturbation theory, such as described in the preceeding subsection, is of course insufficient for many physical purposes (such as the treatment of gravitational radiation reaction for which it is well known to be necessary to go even beyond second order in the post Newtonian approximation). Allowance for actual physical effects involving perturbations at quadratic or higher order is beyond the scope of the present notes. Nevertheless even when one's attention is restricted to first order in so far as "physical" (on shell) perturbations are

concerned, it is still necessary to go to quadratic order in the treatment of "virtual" (off shell) perturbations if one wants a variational formulation of the linearised dynamical equations.

The treatment of higher order perturbations – whether "physical" or "virtual" – is intrinsically straightforeward (though it may of course involve very heavy algebra) when it only involves continuous background fields for which a purely "Eulerian" (effectively fixed) reference system can be used. However when point particles, strings, or higher branes are involved one needs to use a comoving "Lagrangian" type reference system. in the manner developed in Section 2. In so far as only first order perturbation theory is concerned, the necessary adjustment between Eulerian and Lagrangian systems can be treated in accordance with (103) by the well known procedure whereby the corresponding differences are obtained as Lie derivatives with respect to the relevant displacement vector field. However a new issue arises when one comes to generalise such a procedure to quadratic and higher order perturbation theory because in a general mathematical context it is only to first order that a small coordinate displacement will unambiguously characterise and be characterised by a corresponding vector.

Fortunately for what follows, the physical context with which the present notes are concerned is not of the most general mathematical kind, but one in which the background under consideration is endowed with a preferred connection, $\Gamma_{\nu}{}^{\mu}{}_{\rho}$ say, namely the one derived according to the standard Riemannian prescription (20) from the spacetime metric $g_{\mu\nu}$. So long as one has a preferred connection (whether Riemannian or not) to work with, there will after all be a corresponding precription, namely the one naturally specified by the corresponding affinely parametrised geodesic, which we shall refer to simply as "straight" transportation, whereby even to second and higher order, a given vector will unambiguously determine a corresponding displacement and also (at least provided the displacement is not unduly large) vice versa.

To be explicit, let ξ^μ be the components of a vector at a given position with coordinates x^μ, and let ϵ be the corresponding affine parameter along the geodesic starting from x^μ for which the initial affinely normalised tangent vector is ξ^μ. This will characterise a well defined displacement $x^\mu \mapsto x^{\mu}_{\{1\}}$ that is specified by setting $\epsilon = 1$ in the solution $x^{\mu}_{\{\epsilon\}}$ with initial conditions $x^{\mu}_{\{0\}} = x^\mu$, $\dot{x}^{\mu}_{\{0\}} = \xi^\mu$ of the affinely parameterised geodesic equation

$$\ddot{x}^{\mu}_{\{\epsilon\}} + \Gamma_{\{\epsilon\}\nu}{}^{\mu}{}_{\rho}\, \dot{x}^{\mu}_{\{\epsilon\}} \dot{x}^{\rho}_{\{\epsilon\}} = 0\,, \tag{134}$$

where a dot denotes the derivative with respect to the affine parameter. The required solution of this geodesic equation is conveniently expressible by a Taylor expansion whose expression to third order will have the form

$$x^{\mu}_{\{\epsilon\}} - x^\mu = \epsilon\, \$x^\mu + \frac{\epsilon^2}{2!}\$^2 x^\mu + \frac{\epsilon^3}{3!}\$^3 x^\mu + \mathcal{O}\{\epsilon^4\}\,, \tag{135}$$

using a systematic notation scheme whereby the symbol \$ is introduced for the purpose of denoting *differentiation with respect to the affine parameter* at

its initial value, $\epsilon = 0$, so that for example we shall have the abbreviation $\$x^\mu = \dot{x}^{\ \mu}_{\{0\}}$. Proceeding by successive approximations, it is a straightforward exercise to verify that, up to third order, the required coefficients will be given by

$$\$x^\mu = \xi^\mu\,, \qquad\quad \$^2x^\mu = -\Gamma_{\lambda\ \nu}^{\ \mu}\xi^\lambda\xi^\lambda\,, \tag{136}$$

$$\$^3x^\mu = \big(\Gamma_{(\lambda\ \nu)}^{\ \ \mu}\Gamma_{\rho\ \sigma}^{\ \lambda} - \Gamma_{\nu\ \rho,\sigma}^{\ \mu}\big)\xi^\nu\xi^\rho\xi^\sigma\,. \tag{137}$$

The purpose of this scheme is to obtain a systematic calculus that can be used to obtain the generalisation to higher orders of the standard formula (103) whereby starting from a first order Eulerian variation operator $\underset{\text{E}}{\delta}$ the corresponding Lagrangian variation operator $\underset{\text{L}}{\delta}$ as specified with respect to a displaced reference system will be given by $\underset{\text{L}}{\delta} - \underset{\text{E}}{\delta} = \pounds$ where $\pounds$ is the operator of Lie differentiation with respect to the vector ξ^μ that characterises the relative displacement. Starting from a finite Eulerian variation operator $\underset{\text{E}}{\Delta}$ allowing for higher order corrections obtained by successive applications of the corresponding infinitesimal differential operator d_{L} via an Taylor expansion of the form $\underset{\text{E}}{\Delta} = \underset{\text{E}}{\delta} + {}^1/_2\underset{\text{E}}{\delta}^{\,2} + ...$ we want to be able to construct the corresponding finite Lagrangian variation operator $\underset{\text{L}}{\Delta} = \underset{\text{L}}{\delta} + {}^1/_2\underset{\text{L}}{\delta}^{\,2} + ...$ as specified with respect to a reference system that has been subject to a relative transportation operation representable in terms of local coordinates as the mapping $x^\mu \mapsto x^{\ \mu}_{\{1\}}$. By expressing this mapping in terms of the displacement vector ξ^μ in the manner that has just beeen described, the required difference will be obtainable as a Taylor expansion in powers of the vector ξ^μ, and of its first (but not higher) order gradient tensor $\nabla_\nu\xi^\mu$, in the form

$$\underset{\text{L}}{\Delta} - \underset{\text{E}}{\Delta} = \$ + \frac{1}{2}\$^2 + \frac{1}{6}\$^3 + ... \tag{138}$$

where $\$$, $\$^2$, and so on are the relevant operators of first, second, and higher order "straight" differentiation with respect to the ξ^μ. The way to work out the explicit form of the action of such "straight" differentiation operators on the fields of the kind that is of interest – of which the most important is of course the metric $g_{\mu\nu}$ – will be described in the following subsection.

Before proceeding, it is to be observed that the geodesic or "straight" transportation operation, based in the manner just described on the solution of the second order equation (134), needs to be distinguished from the corresponding Lie transportation operation, which is instead based on the solution of a first order equation of the form $\dot{x}^{\ \mu}_{\{\epsilon\}} = \xi^{\ \mu}_{\{\epsilon\}}$. It would in principle be possible to develop higher order displacement perturbation theory in terms of Lie transportation, but the disadvantage of such an aproach is is that (as the price of avoiding dependence on the preferred connection) it would introduce an inordinately high degree of gauge dependence, since it would depend on the specification of the vector ξ^μ not just at the initial point but all the way along the trajectory of the displacement. However the distinction arises only at quadratic and higher order: at first order there is no difference between the "straight" differentiation operator denoted by $\$$, that has just been defined and the corresponding Lie differentiation operation as customarily denoted by the symbol $\pounds$.

It is to be remarked by the way that, just as is customary to use the more explicit notation $\vec{\xi}\pounds$ or alternatively $\pounds_\xi$ whenever it is necessary to indicate the particular vector field ξ^μ with respect to which the Lie differentiation operation $\pounds$ is defined, so analogously we can use the more explicit notation $\vec{\xi}\$$ or alternatively $\$_\xi$ to indicate the particular vector field ξ^μ with respect to which the "straight" differentiation operation $\$$ is defined.

3.3 *"Straight" differentiation of covariant fields

The "straight" displacement scheme set up in the preceeding subsection will evidently determine corresponding transformations of any continuous fields that may be given on the background space. The simplest case is that of an ordinary scalar field Φ say, for which the (affinely parametrised) displacement $x^\mu \mapsto x^{\ \mu}_{\{\epsilon\}}$ will determine a corresponding pullback mapping $\Phi \mapsto \Phi_{\{\epsilon\}}$ with $\Phi_{\{\epsilon\}}\{x\} = \Phi\{x_{\{\epsilon\}}\}$. Evaluating this using an expansion of the form

$$\Phi_{\{\epsilon\}} = \Phi + \Phi_{,\mu}(x^{\ \mu}_{\{\epsilon\}} - x^\mu) + \frac{1}{2}\Phi_{,\mu\nu}(x^{\ \mu}_{\{\epsilon\}} - x^\mu)(x^{\ \nu}_{\{\epsilon\}} - x^\nu) + \mathcal{O}\{\epsilon^3\}\,, \qquad (139)$$

it can be seen from (135) and (137) that up to second order the required result will be given by an expansion of the standard form

$$\Phi_{\{\epsilon\}} = \Phi + \epsilon\,\$\Phi + \frac{\epsilon^2}{2}\$^2\Phi + \mathcal{O}\{\epsilon^3\} \qquad (140)$$

in which the relevant "straight" derivative coefficients are given by

$$\$\Phi = \xi^\mu\nabla_\mu\Phi\,, \qquad \$^2\Phi = \xi^\mu\xi^\nu\nabla_\mu\nabla_\nu\Phi\,. \qquad (141)$$

It is to be noticed that the formulae obtained in this way are of a strictly tensorial nature, unlike those in (137), whose non tensorially coordinate dependent nature is due to that of the coordinates to which they apply. It is also to be noticed that the formula for the second order "straight" derivative $\$^2\Phi$ does not involve derivation of the displacement vector field ξ^μ, unlike the formula for the corresponding second order Lie derivative, which is given by the expression $\pounds^2\Phi = \xi^\mu\nabla_\mu\big(\xi^\nu\nabla_\nu\Phi\big) = \$^2\Phi + (\xi^\mu\nabla_\mu\xi^\nu)\nabla_\nu\Phi$.

After having thus dealt with the case of a scalar, the next simplest case to be considered is that of what Cartan would call a one-form, meaning just a covector, A_μ say, such as the electromagnetic gauge potential, for which the displacement $x^\mu \mapsto x^{\ \mu}_{\{\epsilon\}}$ will determine a corresponding pullback mapping $A_\mu \mapsto A_{\{\epsilon\}\mu}$ with $A_{\{\epsilon\}\mu}\{x\} = A_\nu\{x_{\{\epsilon\}}\}x^{\ \nu}_{\{\epsilon\},\mu}$. In order to work this out, we need to evaluate the relevant tensor transformation matrix $x^{\ \nu}_{\{\epsilon\},\mu}$ whose components are the partial derivatives of the displaced coordinates with respect to the initial coordinates, so that they will be given by an expansion that is obtainable from (134) by partial differentiation of the coefficients (137) in the form

$$x^{\ \mu}_{\{\epsilon\},\lambda} = \delta^\mu_\lambda + \epsilon\,\xi^\mu_{\ ,\lambda} - \frac{\epsilon^2}{2}\big(\Gamma_\nu{}^\mu{}_{\rho,\lambda}\xi^\nu\xi^\rho + 2\Gamma_\nu{}^\mu{}_\rho\xi^{(\nu}\xi^{\rho)}{}_{,\sigma}\big) + \mathcal{O}(\epsilon^3)\,, \qquad (142)$$

in which it is of course to be understood that the fields and their partial coordinate derivatives indicated by the use of a comma, are all to be evaluated as the corresponding functions of the initial (undisplaced) coordinates x^μ. The covectorial analogue of the scalar "straight" transportation formula (139) is thus obtainable in the standard form

$$A_{\{\epsilon\}\mu} = A_\mu + \epsilon\,\$A_\mu + \frac{\epsilon^2}{2}\$^2 A_\mu + \mathcal{O}\{\epsilon^3\} \tag{143}$$

with the relevant "straight" derivative coefficients given by

$$\$A_\mu = \xi^\nu\nabla_\nu\, A_\mu + A_\nu\nabla_\mu\xi^\nu\,, \tag{144}$$

$$\$^2 A_\mu = \xi^\nu\xi^\rho(\nabla_\rho\nabla_\nu A_\mu - \mathcal{R}_{\mu\nu}{}^\lambda{}_\rho A_\lambda) + 2\xi^\nu(\nabla_\mu\xi^\rho)\nabla_\nu A_\rho\,, \tag{145}$$

where $\mathcal{R}_{\mu\nu}{}^\lambda{}_\rho$ is the Riemann tensor associated with the chosen connection, which is definable, using the (MTW) sign convention, by

$$2\nabla_{[\mu}\nabla_{\nu]}A_\rho = -\mathcal{R}_{\mu\nu}{}^\lambda{}_\rho A_\lambda\,. \tag{146}$$

The cancelling out of all the non-tensorially coordinate dependent contributions so as to give a strictly tensorial result is of course not an accident but an automatic consequence of the covariant character of the definition of the straight differentation procedure.

The next case that is of interest in the context of the kind of variational analysis that has been discussed in preceeding sections, is that of a two-index covariant tensor, $B_{\mu\nu}$, which, if it were antisymmetric, as in the example of the Kalb-Ramond potential that was considered in previous work, would be interpretable as a two-form in the sense of Cartan. Whether or not it has such an antisymmetry property, the effect on any two-index covariant tensor of the"straight" transportation $x^\mu \mapsto x^\mu_{\{\epsilon\}}$ will, as in the previous case, be to induce a corresponding pullback mapping $B_{\mu\nu} \mapsto B_{\{\epsilon\}\mu\nu}$ which will be given by $B_{\{\epsilon\}\mu\nu}\{x\} = B_{\rho\sigma}\{x_{\{\epsilon\}}\}x^\rho_{\{\epsilon\},\mu}x^\sigma_{\{\epsilon\},\nu}$. Working this out, using (134), (137), and (142) as in the previous single index case, one can obtain the corresponding expansion in the standard form

$$B_{\{\epsilon\}\mu\nu} = B_{\mu\nu} + \epsilon\,\$B_{\mu\nu} + \frac{\epsilon^2}{2}\$^2 B_{\mu\nu} + \mathcal{O}\{\epsilon^3\} \tag{147}$$

in which the required "straight" derivative coefficients work out as

$$\$B_{\mu\nu} = \xi^\rho\nabla_\rho + B_{\rho\nu}\nabla_\mu\xi^\rho + B_{\mu\rho}\nabla_\nu\xi^\rho \qquad , \tag{148}$$

$$\$^2 B_{\mu\nu} = \xi^\rho\xi^\sigma\big(\nabla_\rho\nabla_\sigma B_{\mu\nu} - \mathcal{R}_{\mu\rho}{}^\lambda{}_\sigma B_{\lambda\nu} - \mathcal{R}_{\nu\rho}{}^\lambda{}_\sigma B_{\mu\lambda}\big) \tag{149}$$

$$+\, 2\xi^\rho\big((\nabla_\mu\xi^\lambda)\nabla_\rho B_{\lambda\nu} + (\nabla_\nu\xi^\lambda)\nabla_\rho B_{\mu\lambda}\big) + 2B_{\rho\sigma}(\nabla_\mu\xi^\rho)\nabla_\nu\xi^\sigma\,. \tag{150}$$

The most important application of the preceeding result is to the case in which $B_{\mu\nu}$ is to taken to be the spacetime metric $g_{\mu\nu}$ itself, for which one thus obtains

$$g_{\{\epsilon\}\mu\nu} = g_{\mu\nu} + \epsilon\,\$g_{\mu\nu} + \frac{\epsilon^2}{2}\$^2 g_{\mu\nu} + \mathcal{O}\{\epsilon^3\} \tag{151}$$

in which the relevant straight derivatives are given by the comparitively simple looking formulae

$$\$ g_{\mu\nu} = 2\nabla_{(\mu}\xi_{\nu)} \ , \qquad \$^2 g_{\mu\nu} = 2(\nabla_{(\mu}\xi^\rho)\nabla_{\nu)}\xi_\rho - 2\xi^\rho\xi^\sigma \mathcal{R}_{\mu\rho\nu\sigma} \, , \tag{152}$$

of which the former is the same as the usual Lie differentiation formula while the latter agrees with a result that was obtained in a different context by Boisseau and Letelier [64].

3.4 *Perturbed Dirac Goto Nambu action

The calculus developed in the preceding sections is intended for application to cases in which one is concerned with a perturbed configuration that is obtainable from a given unpertubed perturbation by a small deviation, of order δ say, via a continuously differentiable (homotopic) variation, which may be taken to be specified by a homotopy parameter λ say ranging from 0 in the unperturbed configuration to 1 in the particular perturbed configuration under consideration. Using an overhead prime to denote differentiation with respect to λ the "straight" transport vector ξ^μ characterising the relative displacement of the final perturbed configuration (for which $\lambda = 1$) will be expressible by an expansion of the form

$$\xi^\mu = \acute{\xi}^\mu + \tfrac{1}{2}\acute{\acute{\xi}}^\mu + \mathcal{O}\{\delta^3\} \, . \tag{153}$$

Specifying the differential operator d with respect to the same parameter λ, so that for example in the case of the action $\mathcal{I}$ we obtain the identification $\delta\mathcal{I} = \acute{\mathcal{I}}$, the operator Δ giving the deviation between the original and perturbed value of any additive quantity will be expressible by a corresponding expansion

$$\Delta = \delta + \tfrac{1}{2}\delta^2 + \mathcal{O}\{\delta^3\} \, . \tag{154}$$

The reason why we are interested in going to second order in the present notes is that – according to a very general principle first drawn to my attention many years ago by Taub [42], [43] – if a system of dynamical equations is obtainable by a variational principle from an action $\mathcal{I}$ (meaning that the dynamical equations are the condition for the first order locally perturbed action $\delta\mathcal{I}$ to vanish) then the corresponding first order perturbed dynamical equations will be similarly obtainable from the *second* order perturbed action $\delta^2\mathcal{I}$.

For a free brane brane action of the kind governed by (131), the first order perturbed action with respect to a Lagrangian reference system (meaning one that is comoving with the brane worldsheet) is directly obtainable (just by setting the external field variables A_μ and $B_{\mu\nu}$ set to zero) from (88) in the form

$$\delta\mathcal{I} = \frac{1}{2}\int d\overline{\mathcal{S}}\ \overline{T}^{\mu\nu} \underset{\mathrm{L}}{\delta} g_{\mu\nu} \, , \tag{155}$$

and so the corresponding second order perturbed action will be obtainable, using (91), in the form

$$\delta^2\mathcal{I} = \frac{1}{2}\int d\overline{\mathcal{S}} \left((\underset{\mathrm{L}}{\delta}\overline{T}^{\mu\nu})\underset{\mathrm{L}}{\delta} g_{\mu\nu} + \overline{T}^{\mu\nu}\big(\tfrac{1}{2}\eta^{\rho\sigma}(\underset{\mathrm{L}}{\delta} g_{\rho\sigma})\underset{\mathrm{L}}{\delta} g_{\mu\nu} + \underset{\mathrm{L}}{\delta}{}^2 g_{\mu\nu}\big)\right) . \tag{156}$$

In the case to which our attention will be restricted here, the background metric will be taken to be held fixed in the sense that its Eulerian variation vanishes, $\underset{\rm E}{\Delta} g_{\mu\nu} = 0$. This means that according to the general displacement rule (138) the Lagrangian (comoving) metric variation in which we are interested will be given by

$$\underset{\rm L}{\Delta} = \vec{\xi}\$ + \tfrac{1}{2}\big(\vec{\xi}\$\big)^2 + \mathcal{O}\{\delta^3\}\,. \tag{157}$$

The concise abbreviation \$ for the straight differentation operator involved in (138) has been replaced here by the more explicit notation $\vec{\xi}\$$ to indicate its dependence on the displacement vector whose expansion is given by (153). It can thus be seen that the first and second order Lagrangian variations of the metric will be given by

$$\underset{\rm L}{\delta} g_{\mu\nu} = \vec{\xi}'\$ g_{\mu\nu}\ , \qquad\qquad \underset{\rm L}{\delta}^2 g_{\mu\nu} = \big(\vec{\xi}'\$\big)^2 g_{\mu\nu} + \vec{\xi}''\$ g_{\mu\nu}\,. \tag{158}$$

When the relevant unperturbed field equation, namely (131), are satisfied, the first order action variation due to any perturbation with compact support will of course vanish by the variation principle, $\delta^2\mathcal{I} = 0$. In these circumstances the contribution from the second order displacement contribution ξ''^{μ} will automatically cancel out, so that for the Dirac-Goto-Nambu model determined by a surface Lagrangian of the trivial constant form

$$^{(\rm d)}\overline{\mathcal{L}} = -m^{\rm d} \tag{159}$$

the second order action variation can be obtained from (156), using the formula (152) in the form

$$\delta^2\mathcal{I} = \int \frac{1}{2}\int d\overline{\mathcal{S}}\,\overline{T}^{\mu}{}_{\nu}\Big(\perp_{\rho\sigma}\big(\overline{\nabla}_{\mu}\xi'^{\rho}\big)\overline{\nabla}^{\nu}\xi'^{\sigma} + 2\big(\overline{\nabla}_{[\rho}\xi'^{\rho}\big)\overline{\nabla}_{\mu]}\xi'^{\nu} - \mathcal{R}_{\mu\rho}{}^{\nu}{}_{\sigma}\xi'^{\rho}\xi'^{\sigma}\Big)\,. \tag{160}$$

with

$$^{(\rm d)}\overline{T}^{\mu\nu} = -m^{\rm d}\,{}^{(\rm d)}\eta^{\mu\nu}\,. \tag{161}$$

where m is a constant having the dimension of mass (which would be of the order of magnitude of the relevant Higgs mass scale in the case of a vacuum defect arising from the spontaneous symmetry breaking mechanism of the kind most commonly considered [6]) and d is the dimension of the worldsheet (i.e. d=1 for a simple point particle, d=2 for a string, and so on).

Substituting the expression (161) for the surface stress energy momentum tensor into (131), the unperturbed Dirac Gotu Nambu equation of motion is obtained in a form that is given – independently of the dimension d indicated by the prefix $^{(\rm d)}$ which may therefore be dropped – by the well known harmonicity condition that is expressible as the vanishing of the curvature vector,

$$K^{\mu} = 0\,. \tag{162}$$

In this rather degenerate special case, since there are no internal fields the linearised perturbation equation (132) will of course completely determine the

evolution of the displacements by itself. By substituting (161) in (132) the corresponding perturbation equation is obtained in the form

$$\perp^{\rho}_{\lambda}\big(\overline{\nabla}^{\mu}\overline{\nabla}_{\mu}\xi^{\lambda}-\eta^{\mu\nu}\mathcal{R}^{\lambda}{}_{\mu\nu\sigma}\xi^{\sigma}\big)=2K_{\mu\nu}{}^{\rho}\,\overline{\nabla}^{\mu}\xi^{\nu}\,. \tag{163}$$

It is straightforeward to check that this does indeed agree – subject to (162) – with what is obtained [44] by applying the variation principle to the second order action (160). As well as the compact tensorial version (163) given here, the literature includes other equivalent but formally more complicated expressions [45], [46] (involving reference to internal coordinates or surface adapted frames of the kind discussed in Subection 1.1) that generalise earlier work which was restricted to a flat or De Sitter background [47], [48], [48] or specialised to the hypersurface supported ("wall" or membrane) special case [50].

3.5 *Higher order geodynamic models

For most practical physical purposes the most useful generalisations of the Dirac-Goto-Nambu models governed by (162) and (163) are those of the very general category governed by (131) and (132) which allow for internal fields such as the currents that can be used, not only to represent the Witten type superconductivity effect in cosmic strings, but also also to represent the effect of ordinary elasticity in terrestrial applications, such as the strings of musical instruments which were already a subject of scientific investigation, albeit at an empirical rather than theoretical level, in the time of Pythagoras. However before proceeding to the discussion of such internal field effects in the following sections, it is of interest to consider how the simple Dirac-Goto-Nambu model can be generalised in a way that goes beyond the "pure" brane description characterised in the free case by (131) and in the presence of an external force by (122).

The distinguishing property of a "pure" brane model of the kind considered in Section 2 is the condition that the action depends only on the undifferentiated background fields $g_{\mu\nu}$, A_{μ}, $B_{\mu\nu}$, but not on their gradients. One of the most familiar kinds of example in which this condition fails to hold is that of an electromagnetically polarised medium, whose action [51] depends not just on the gauge field A_{μ} but also directly on the associated field $F_{\mu\nu}$ itself. For lack of time and space we shall not consider such electromagnetic effects here, but will consider only the simplest category of "geodynamic" brane models, meaning those in which, as in the "pure" brane models of the Dirac-Goto-Nambu category, the action depends only on the imbedding geometry of the worldsheet and not on any other external internal fields. The simplest such extension of the "pure" Dirac-Goto-Nambu model (whose action is proportional just to the surface measure which depends only on $g_{\mu\nu}$ but not its derivatives) is based on a Lagrangian consisting not only of a constant term but also of terms proportional to the two independent scalars that can be constructed as quadratic functions of the first derivatives of the metric, namely $K_{\mu}K^{\mu}$ and $K_{\mu\nu\rho}K^{\mu\nu\rho}$. The inclusion of such "stiffness" terms has been suggested by Polyakov [52] and others, one of the main reasons being allowance for the deviations from the "pure" Dirac-Goto-Nambu description of cosmological string [53], [54], [55] or domain wall

[56], [57], [58], [59], [60] defects that one would expect to arise if the curvature becomes too strong. The kind of Lagrangian constructed in this way, namely

$$ {}^{(\mathrm{d})}\overline{\mathcal{L}} = -m^{\mathrm{d}} + \mathrm{b}\,{}^{(\mathrm{d})}K_{\rho}\,{}^{(\mathrm{d})}K^{\rho} - \mathrm{c}\,{}^{(\mathrm{d})}K_{\mu\nu\rho}\,{}^{(\mathrm{d})}K^{\mu\nu\rho}\,, \tag{164} $$

where m, b, and c are constants, has recently been the subject of investigations by a number of authors [61], [62], [63], [64], [65], [66]. It is convenient to use the abbreviation

$$ {}^{(\mathrm{d})}\Sigma^{\mu\nu\rho} = \mathrm{b}\,{}^{(\mathrm{d})}\eta^{\mu\nu}\,{}^{(\mathrm{d})}K^{\rho} - \mathrm{c}\,{}^{(\mathrm{d})}K^{\mu\nu\rho}\,, \tag{165} $$

which enables the Lagrangian (164) to be expressed in condensed form as

$$ {}^{(\mathrm{d})}\overline{\mathcal{L}} = -m^{\mathrm{d}} + {}^{(\mathrm{d})}\Sigma_{\mu\nu\rho}\,{}^{(\mathrm{d})}K^{\mu\nu\rho}\,. \tag{166} $$

The variation needed for evaluating the change in such an "impure" brane action will thereby be obtainable from the formulae above in the corresponding form

$$ \|\gamma\|^{-1/2}\underset{\mathrm{L}}{\delta}\big(\|\gamma\|^{1/2}\overline{\mathcal{L}}\big) = \underset{\mathrm{L}}{\delta}\overline{\mathcal{L}} + \frac{1}{2}\overline{\mathcal{L}}\,\eta^{\mu\nu}\underset{\mathrm{L}}{\delta}g_{\mu\nu} \tag{167} $$

(again dropping the explicit reference to the brane dimension d) with

$$ \underset{\mathrm{L}}{\delta}\overline{\mathcal{L}} = \big(\Sigma^{\lambda\rho\mu}K_{\lambda\rho}{}^{\nu} - 2\,\Sigma^{\lambda\mu}{}_{\rho}K_{\lambda}{}^{\nu\rho}\big)\underset{\mathrm{L}}{\delta}g_{\mu\nu} + \big(2\,\Sigma^{\mu\lambda\nu} - \Sigma^{\mu\nu\lambda}\big)\nabla_{\lambda}\underset{\mathrm{L}}{\delta}g_{\mu\nu}\,. \tag{168} $$

Although it is still possible to construct a formally symmetric stress momentum energy density tensor of the distributional type, the presence of the gradient term on the right of (168) will make it rather pathological, with not just a contribution proportional to a Dirac distribution as in the "pure" brane case described by (114) but also with a contribution proportional to the even more highly singular gradient of a Dirac distribution [64]. In order to be able to continue working in terms of strictly regular surface supported field, it is necessary [65] to deal with the gradient dependence of the action in such an "impure" brane model by having recourse to the use of a total stress momentum energy tensor $\mathcal{T}^{\mu}{}_{\nu}$ of the no longer no longer symmetric canonical type that can be read out from the variation formula

$$ \|\gamma\|^{-1/2}\underset{\mathrm{L}}{\delta}\big(\|\gamma\|^{1/2}\overline{\mathcal{L}}\big) = \mathcal{T}^{\mu}{}_{\nu}\overline{\nabla}_{\mu}\xi^{\nu} + 2\Sigma_{\mu}{}^{\rho\sigma}\,\mathcal{R}_{\rho\sigma}{}^{\mu}{}_{\nu}\xi^{\nu} - \overline{\nabla}_{\mu}\big(2\,\Sigma_{\nu}{}^{\mu\rho}\nabla_{\rho}\xi^{\nu}\big) \tag{169} $$

that is obtained after substitution of (126) in (167) and (168). The regular but non-symmetric *canonical surface stress momentum energy density* tensor $\mathcal{T}^{\mu}{}_{\nu}$ obtained in this way [65] is given by

$$ \mathcal{T}^{\mu}{}_{\nu} = \widetilde{T}^{\mu}{}_{\nu} + \bar{t}^{\mu}{}_{\nu}\,, \tag{170} $$

where $\widetilde{T}^{\mu\nu}$ is symmetric and purely tangential to the worldsheet (as a regular "geometric" stress momentum energy density tensor would be) with a merely algebraic dependence on the second fundamental tensor, being given by

$$ \widetilde{T}^{\mu}{}_{\nu} = \widetilde{T}_{\nu}{}^{\mu} = \mathcal{T}^{\mu}{}_{\lambda}\eta^{\lambda}_{\ \nu} = \overline{\mathcal{L}}\,\eta^{\mu}_{\ \nu} - 2\,\Sigma^{\lambda}{}_{\nu\rho}\,K_{\lambda}{}^{\mu\rho}\,, \tag{171} $$

while the remainder, which is of higher differential order, is expressible in terms of the third fundamental tensor (24) as

$$\bar{t}^\mu{}_\nu = \mathcal{T}^\mu{}_\lambda \perp^\lambda_\nu = 2\,\mathrm{c}\,\Xi^\lambda{}_\lambda{}^\mu{}_\nu - 2\,\mathrm{b}\,\Xi^\mu{}_\lambda{}^\lambda{}_\nu\,, \tag{172}$$

which can conveniently be rewritten with the higher derivative contributions regrouped in the form

$$\bar{t}^\mu{}_\nu = 2(\mathrm{c}-\mathrm{b})\Xi^\mu{}_\nu + \mathrm{c}\,\eta^\mu{}_\rho\eta^{\sigma\tau}\mathcal{R}^\rho{}_{\sigma\tau\lambda}\perp^\lambda_\nu\,, \tag{173}$$

where

$$\Xi^\mu{}_\nu = \Xi^\mu{}_\lambda{}^\lambda{}_\nu = \perp^\lambda_\nu \overline{\nabla}^\mu K_\lambda\,. \tag{174}$$

It is to be noticed that the total canonical surface stress momentum energy density tensor obtained in this way is is still automatically tangential to the worldsheet on its first (though no longer on its second) index, i.e.

$$\perp^\lambda_\mu \mathcal{T}^\mu{}_\nu = 0\,, \tag{175}$$

and that the higher derivative contribution proportional to the trace $\Xi^\mu{}_\nu$ of the third fundamental tensor will drop out if the coefficients have the same value, $\mathrm{c}=\mathrm{b}$.

The application of the variation principle to the effect that the surface integral of the variation (169) should vanish for any displacement ξ^μ within a bounded neighbourhood can be seen to lead (via an application of Green's theorem as in the Section 2 to dynamical equations [65] of the form

$$\overline{\nabla}_\mu \mathcal{T}^\mu{}_\nu = 2\,\Sigma_\mu{}^{\rho\sigma}\,\mathcal{R}_{\rho\sigma}{}^\mu{}_\nu\,. \tag{176}$$

As in the pure Dirac-Gotu-Nambu case discussed above, the foregoing system of equations is partially redundant: although it involves n distinct spacetime vectorial equations, only n–d of them are dynamically independent, namely those projected orthogonally to the d-dimensional worldsheet. The others are merely Noether identities which follow independently of the variation principle from the fact that a displacement ξ^ν that is purely tangential to the worldsheet merely maps it onto itself and thus cannot affect the action, as can be verified directly using the generalised Codazzi identity (25).

The lack of symmetry of $\mathcal{T}^\mu{}_\nu$ means that the construction of the corresponding momentum current vector, $\mathcal{P}^\mu$ say, associated with a generic background spacetime Killing vector field k^μ will not be quite as simple as in the "pure" brane case for which an expression of the simple form (119) suffices. However using the well known fact that the Killing equation (118) entails the integrability condition

$$\nabla_\mu\nabla_\nu k^\rho = \mathcal{R}^\rho{}_{\nu\rho\lambda}k^\lambda\,, \tag{177}$$

together with the observation that the antisymmetric part of the canonical stress momentum energy density tensor (171) is given according to (172) just by

$$\mathcal{T}^{[\mu\nu]} = -2\overline{\nabla}_\lambda \Sigma^{\lambda[\mu\nu]}\,, \tag{178}$$

it can be seen that, for any solution of (118), the ansatz [65]

$$\mathcal{P}^\mu = \mathcal{T}^\mu{}_\nu k^\nu + 2\Sigma^{\mu\nu}{}_\rho \nabla_\nu k^\rho \tag{179}$$

provides a surface current, $\mathcal{P}^\mu$, which satisfies the tangentiality condition

$$\perp^\lambda_\mu \mathcal{P}^\mu = 0\,, \tag{180}$$

and for which the strict surface conservation law,

$$\overline{\nabla}_\mu \mathcal{P}^\mu = 0\,, \tag{181}$$

will hold whenever the equation of motion (176) is satisfied. It is to be remarked however that for a Killing vector of the irrotational kind for which $\nabla_\mu k_\mu$ vanishes altogether the second term (interpretable as a surface spin density contribution) in (179) will not contribute, i.e. an expression of the simpler form (119) will suffice. This applies in particular to the case of an ordinary translation generator in flat space, for which the corresponding conserved surface current will represent ordinary energy or linear momentum, whereas in the case of angular momentum the extra (spin density) term in (179) is indispensible.

The preceeding formulae all include allowance for arbitrary background curvature, but, to obtain the analogue of the non redundant version (131) of the equations of motion in a reasonably simple form, the restriction that the background spacetime be flat, i.e. $\mathcal{R}^\mu{}_{\nu\rho\sigma} = 0$, will now be imposed. This enables the required system of dynamical equations to be expressed [65] in the form

$$\widetilde{T}^{\mu\nu} K_{\mu\nu\rho} = 2(\mathrm{b}-\mathrm{c}) \perp^\nu_\rho \overline{\nabla}_\mu \big(\perp^\sigma_\nu \overline{\nabla}^\mu K_\sigma\big)\,, \tag{182}$$

with the higher derivative terms grouped on the right hand side, which vanishes if $\mathrm{b} = \mathrm{c}$.

In the particular case [60] of a membrane, meaning a brane supported by a hypersurface of dimension d=n–1, the second fundamental tensor and its trace will be given in terms of the unit normal λ_ρ (which in this case will be unique up to a choice of sign) by $K_{\mu\nu\rho} = K_{\mu\nu}\lambda_\rho$ and $K_\rho = K\lambda_\rho$ with $K = K^\nu{}_\nu$ where $K_{\mu\nu}$ is the second fundamental form (whose sign depends on that orientation chosen for the normal). In an ordinary 4-dimensional spacetime background, this membrane case corresponds to d=3, for which the symmetric tangential part of the surface stress momentum energy density tensor will be therefore be obtainable from (171) in the form

$$^{(3)}\widetilde{T}^{\mu\nu} = -\big(m^3 - \mathrm{b}\,{}^{(3)}\!K^2 + \mathrm{c}\,{}^{(3)}\!K^\rho{}_\sigma\,{}^{(3)}\!K^\sigma{}_\rho\big)\,{}^{(3)}\eta^{\mu\nu} - 2\mathrm{b}\,{}^{(3)}\!K\,{}^{(3)}\!K^{\mu\nu} + 2\mathrm{c}\,{}^{(3)}\!K^{\lambda\mu}\,{}^{(3)}\!K_\lambda{}^\nu\,. \tag{183}$$

In the case of a string with d=2 one can use the fact that the trace free conformation tensor (22) will satisfy $2\,{}^{(2)}C^\lambda{}_{\mu\rho}\,{}^{(2)}C_{\lambda\nu}{}^\rho = {}^{(2)}C_{\kappa\lambda\rho}\,{}^{(2)}C^{\kappa\lambda\rho}\,{}^{(2)}\eta_{\mu\nu}$ to obtain a corresponding formula (which holds regardless of the background spacetime dimension n) given [65] by

$$^{(2)}\widetilde{T}^{\mu\nu} = -m^2\,{}^{(2)}\eta^{\mu\nu} + 2(\mathrm{c}-\mathrm{b})\,{}^{(2)}C^{\mu\nu\rho}\,{}^{(2)}K_\rho\,. \tag{184}$$

4 Strings and other spacially isotropic brane models

4.1 The general category of "perfect" brane models

It is reasonable to postulate that a "weak" energy condition of the kind formulated and justified by Hawking and Ellis [15] should hold for any pure p-brane model as a condition for physical realism as a macroscopic description of a (p+1)-surface supported physical system at a classical level, meaning that the model's surface stress momentum energy density tensor (as introduced in Section 2) should be such that the contraction $\overline{T}^{\mu\nu}\beta_\mu\beta_\nu$ is non negative for any vector β^μ that is timelike. Furthermore the causality condition to the effect that there should be no timelike characteristic covector (i.e. no superluminal propagation) can be seen from (133) to entail the further requirement (going marginally beyond the "weak" condition of Hawking and Ellis) that $\overline{T}^{\mu\nu}\beta_\mu\beta_\nu$ should be strictly positive if β^μ is timelike. This leads to the formulation of what may be called the "minimal" energy condition for a pure p-brane which is expressible as

$$\beta^\mu\beta_\mu < 0 \qquad \Rightarrow \qquad \overline{T}^{\mu\nu}\beta_\mu\beta_\nu > 0\,. \tag{185}$$

A (pure) p-brane model will consequently be characterised by a well defined *surface energy density*, U say, that is specifiable by an eigenvalue equation of the form

$$\overline{T}^\mu{}_\nu\beta^\nu = -U\beta^\mu\,, \tag{186}$$

where the corresponding eigenvector β^μ is distinguished by the requirement that it be tangential and non-spacelike:

$$\perp^\mu_\nu\beta^\nu = 0\,, \qquad \beta^\mu\beta_\mu \leq 0\,. \tag{187}$$

It is apparent that the "minimal"energy condition (185) requires that the eigenvalue U should be strictly positive unless β^μ is null in which case it may vanish:

$$U \geq 0\,, \qquad U = 0 \qquad \Rightarrow \qquad \beta^\mu\beta_\mu = 0\,. \tag{188}$$

In the Dirac-Goto-Nambu model that is most familiar to present day cosmologists, the eigenvector β^μ is indeterminate and the energy density U is the same (in relativistic units such as are used here, with the speed of light set to unity) as the corresponding (surface) tension. However in general it is essential to distinguish the concept of energy density U from the concept of the *tension* (as used in physics since the formulation of Hooke's law at the time of Newton) from which term "tensor" is derived. The (surface) tension scalar, for which we shall use the traditional symbol T, is defineable generically in a manner consistent with traditional usage, for a "pure" p-brane (i.e. a (p+1)-dimensionally supported system) of the kind considered here by decomposing the trace of the surface stress momentum energy density tensor in the form

$$\overline{T}^\nu_\nu = -U - \mathrm{p}T\ . \tag{189}$$

Apart from the degenerate Dirac-Goto-Nambu case for which U and T are actually equal, the simplest possibility is that of a *perfect* p-brane [5], meaning one whose surface stress momentum energy density tensor is spacially (thus p-dimensionally) *isotropic* so that it is will be expressible, for a suitable choice of the normalisation of the eigenvector β^μ in (186) by

$$\overline{T}^{\mu\nu} = \beta^\mu \beta^\nu - T\eta^{\mu\nu} \,, \tag{190}$$

where the required normalisation is given by

$$U - T = \beta^\mu \beta_\mu \leq 0 \,. \tag{191}$$

This category includes the case of the Dirac-Goto-Nambu model (161), which is obtained (with $T = m^{\mathrm{p}+1}$) by normalising the (in this case indeterminate) eigenvector to zero, i.e setting $\beta^\mu = 0$. A more mundane example is provided by the familiar "improper" case p=n–1 (where n is the background dimension) of an ordinary (relativistic) perfect fluid with pressure $P = -T$. Although the tension T is negative in the ordinary fluid case, it must be positive, as a condition for stability, for "proper" p-branes of lower dimension, p<n–1. Whereas the membrane models (with p=2 for n=4) that are appropriate for the description of nautical sails will not in general have the isotropic form (190), a familiar everyday example of a membrane that will automatically be "perfect" in this sense is provided by the case of an ordinary soap bubble whose boundary hypersurface will be characterised by $0 < T \ll U$. The kind of application on which we shall concentrate in the following sections is that of a *string*, as given by p=1, for which the "perfection" property (190) will always hold: isotropy cannot fail to apply in this 1-brane case because only a single space dimension is involved.

Before specialising to the (automatically perfect) string case, it is to be remarked that except when the β^μ is indeterminate (as in the Dirac-Goto-Nambu case), or null (as can occur in special subspaces where the current becomes null in a string model of the kind [36], [37], [41] appropriate for describing cosmic vortex defects of the "superconducting" kind proposed by Witten [8]) this eigenvector will determine a corresponding unit eigenvector,

$$u^\mu = \big(U - T\big)^{-1/2} \beta^\mu \,, \qquad u^\mu u_\mu = -1 \,, \tag{192}$$

specifying a naturally preferred rest frame, in terms of which (190) will be rewriteable in the standard form

$$\overline{T}^{\mu\nu} = \big(U - T\big) u^\mu u^\nu - T\eta^{\mu\nu} \,. \tag{193}$$

With respect to the frame so defined, it can be seen that the velocity $c_{\rm E}$ say of propagation of extrinsic perturbations (what are commonly referred to as "wiggles") of the perfect brane worldsheet will be given, according to the characteristic equation (133), by

$$c_{\rm E}{}^2 = \frac{T}{U} \,. \tag{194}$$

By substituting (192) in the (125), and using the expression (18) for the acceleration $\dot{u}^\mu$ of the preferred unit vector (192), the extrisic equation governing the free motion of the support surface of any (proper) perfect brane is reducible to the standard form [5], [41]

$$c_{\rm E}{}^2 K^\mu = (1 - c_{\rm E}{}^2) \perp^{\!\mu}_{\nu} \dot{u}^\nu \,. \tag{195}$$

In the Dirac-Goto-Nambu case one has $U = T$ and hence $c_{\rm E}{}^2 = 1$, (i.e. the "wiggle" speed is that of light) so that the right hand side of (195) will drop out, leaving the harmonicity condition that is expressed by the vanishing of the curvature vector given by (21). At the opposite extreme from this relativistic limit case, one has applications to such mundane examples as that of an ordinary soap bubble membrane or violin string for which one has $c_{\rm E}{}^2 \ll 1$.

4.2 The special case of "barotropic" brane models

Except in the Dirac-Goto-Nambu limit case, the extrinsic equation of motion (196) will need to be supplemented by the dynamic equations governing the evolution of the internal fields, and in particular of T, U, and u^μ, on the brane surface. The simplest non trivial possibility is the case of a perfect brane model that is "barotropic", meaning that its tension T (or equivalently, in the case of a fluid, its pressure P) is a function only of the energy density U in the preferred rest frame. In this barotropic or "perfectly elastic" case, the complete set of dynamical equations governing the internal evolution of the perfect brane will be provided just by the tangential projection of the force balance law (110) which, in the free case to which the discussion of the present section is restricted, has the simple form

$$\eta^\lambda{}_\nu \overline{\nabla}_{\!\mu} \overline{T}^{\mu\nu} = 0 \,. \tag{196}$$

In the barotropic case (which includes the kind of cosmic string models [36], [37], [41] appropriate for describing the Witten type superconducting vaccuum vortices [8] whose investigation provided the original motivation for developing the kind of analysis presented here) experience of the "improper" case of an (n–1)-brane, i.e. the familiar example of an ordinary barotropic perfect fluid, suggests the convenience of introducing an idealised particle *number density*, ν say, and a corresponding (relativistic) *chemical potential*, or effective mass per idealised particle, μ say, that are specified modulo an arbitrary normalisation factor by the equation of state according to a prescription of the form

$$\ln \nu = \int \frac{dU}{U - T} \,, \qquad \ln \mu = \int \frac{dT}{T - U} \,, \tag{197}$$

which fixes them modulo a pair of constants of integration which are not allowed to remain independent but are related by the imposition of the restraint condition

$$\mu\nu = U - T \,. \tag{198}$$

The quantities introduced in this way can be used for defining a surface particle current density vector ν^ρ and a corresponding dynamically conjugate energy - momentum (per particle) covector μ_ρ that are given by

$$\nu^\rho = \nu u^\rho \ , \qquad\qquad \mu_\rho = \mu u_\rho \, , \tag{199}$$

in terms of which the generic expression (193) of the surface stress momentum energy density tensor of a perfect brane can be rewritten in the form

$$\overline{T}^\rho{}_\sigma = \nu^\rho \mu_\sigma - T\eta^\rho{}_\sigma \, , \tag{200}$$

whose divergence can be seen to be given identically by

$$\overline{\nabla}_\rho \overline{T}^\rho{}_\sigma = \mu_\sigma \overline{\nabla}_\rho \nu^\rho + 2\nu^\rho \overline{\nabla}_{[\rho}\mu_{\sigma]} - TK_\sigma \, . \tag{201}$$

In the free case for which this divergence vanishes, projection orthogonal to the worldsheet gives back the extrinsic dynamical equation (195), while by contraction with ν^ρ one obtains the simple surface current conservation law

$$\overline{\nabla}_\rho \nu^\rho = 0 \, . \tag{202}$$

The remainder of the system of equations of motion for the free perfect brane can hence be obtained from (196), as a surface generalised version of the standard perfect fluid momentum transport law [67], in the form

$$u^\mu \eta^\rho{}_\nu \overline{\nabla}_{[\mu}\mu_{\rho]} = 0 \, , \tag{203}$$

which is such as to entail a corresponding Kelvin type law of conservation (under Lie transport by the flow) for the relevant surface vorticity tensor $\eta^\rho{}_{[\nu}\overline{\nabla}_{[\mu]}\mu_{\rho]}$ so that if it vanishes initially it will remain zero throughout, which is the integrability condition for the existence of a surface potential function, φ say, such that $\mu_\rho = \overline{\nabla}_\rho \varphi$.

Appart from the extrinsic perturbations of the world sheet location itself, which propagate with the "wiggle" speed $c_{\rm E}$ (relative to the frame deterined by u^μ) as already discused, the only other kind of perturbation mode that can occur in a barytropic string are longitudinal modes specified by the variation of U or equivalently of T within the world sheet. As in the "improper" special case of an ordinary perfect fluid, for which they are interpretable just as ordinary sound waves, such longitudinal "woggle" or "jiggle" perturbations can easily be seen from (200) and (201) to have a relative propagation velocity $c_{\rm L}$ say that is given [5], [41] by

$${c_{\rm L}}^2 = \frac{\nu}{\mu}\frac{d\mu}{d\nu} = -\frac{dT}{dU} \, . \tag{204}$$

In order for a barotropic model to be well behaved, ${c_{\rm E}}^2$ and $c_{\rm L}$ must of course both be positive, in order for the velocities to be real (as a condition for local stability), and they must also both be less than unity in order to avoid superluminal propagation (as a condition for local causality). An important qualitative

question that arises for any *proper* barotropic p-brane, i.e. one with 1≤ p<n–1 (excluding the trivial case p=0 of a point particle, and at the other extreme the "improper" case p=n–1 of a fluid) is whether the extrinsic propagation speed is subsonic, meaning that "wiggles" go slower than "jiggles", i.e $c_{\rm E} < c_{\rm L}$, transonic, meaning that the speeds are the same, $c_{\rm E} = c_{\rm L}$ or supersonic, meaning that "wiggles" go faster than "jiggles", $c_{\rm E} > c_{\rm L}$. Much of the earliest work [68], [69], [70], [71], [72], [73], [74] on the Witten type superconducting vaccuum vortex phenomena was implicitly based on use of a string model of subsonic type, but a more careful analysis of the internal structure of the Witten vortex by Peter [75], [76], [77] has since shown that models of supersonic type are more appropriate. On the other hand such everyday examples as that of an ordinary violin string are subsonic, a condition that is sufficient [78] though not necessary [79], [80] for stability of the corresponding circular centrifugally supported "vorton" type loop configurations [81], [82], [83], [84] that will be discussed later on.

A particularly important special case is that of a barytropic p-brane of *permanently transonic* type, meaning one characterised by an equation of state of the non dispersive constant product form

$$UT = m^{2{\rm p}+2}\,, \tag{205}$$

where m is a constant having the dimensions of a mass, for which application of the formulae (194) and (202) gives ${c_{\rm E}}^2 = {c_{\rm L}}^2$ not just for some critical transition value of U or T (as can occur for other kinds of equation of state) but for all values. In the string case, p=1, it can easily be demonstrated [5] that this non dispersive permanently transonic model represents the outcome of a (rather artificial) dimensional reduction process first suggested by Nielsen [85], [86], [87], [88]. As will be described below, this model can be shown [89] to be governed by equations of motion that are explicitly integrable in a flat empty background, and it can be used [89], [90] to provide what (contrary to a misleading claim [91] that has been made) can in fact [92], [93] be a highly accurate description of the macroscopic effect of "wiggles" in an underlying cosmic string of the simple Goto-Nambu type.

5 Essentials of elastic string dynamics

5.1 *Bicharacteristic formulation of extrinsic equations

Between the hypersurface supported case of a membrane and the curve supported case of a point particle, the only intermediate kind of brane that can exist in 4-dimensions is that of 1-brane, i.e. a string model, which (for any background dimension n) will have a first fundamental tensor that is expressible as the square of the antisymmetric unit surface element tensor $\mathcal{E}^{\mu\nu} = -\mathcal{E}^{\nu\mu}$ given by (75):

$$\eta^{\mu}{}_{\nu} = \mathcal{E}^{\mu}{}_{\rho}\mathcal{E}^{\rho}{}_{\nu}\,. \tag{206}$$

In the case of string (to which the remainder of this article will be devoted) the symmetric surface stress momentum energy tensor $\overline{T}^{\mu\nu}$ that is well defined for any "pure" (i.e. unpolarised) model will not only be expressible in the generic form (190) that expresses the (in the string case trivial) property of spacial anisotropy with respect to the preferred timelike or null eigenvector β^μ, but it will also be expressible generically in *bicharacteristic form* as

$$\overline{T}^{\mu\nu} = \beta_+^{[\mu}\beta_-^{\nu]}\,, \tag{207}$$

in terms of a pair of timelike or null tangent vectors $\beta_\pm^\mu$ that can be seen to be extrinsic bicharacteric vectors, meaning that they lie respectively along the two ("right moving" and "left moving") directions of propagation of extrinsic perturbations, and are thus respectively orthogonal to the covectors χ_μ given by the extrinsic characteristic equation (133) whose solutions can be seen from (207) to be given by $\beta_+^\mu\chi_\mu = 0$ or $\beta_-^\mu\chi_\mu = 0$. The expression (207) does not completely determine the bicharacteristic vectors $\beta_\pm^\mu$, but leaves open the possibility of a reciprocal multiplicative rescaling of their magnitudes, subject to the condition that their scalar product remains invariant, its value being obtainable by comparison of (207) with (190) as

$$\beta_{+\mu}\beta_-^\mu = \overline{T}^\nu{}_\nu = -\big(U+T\big)\,. \tag{208}$$

The geometric mean of the magnitudes of the bicharacteristic vectors remains similarly invariant, with value given by the magnitude of the preferred eigenvector β^μ as given by (191), i.e.

$$\big(\beta_{+\mu}\beta_+{}^\mu\big)\big(\beta_{-\nu}\beta_-{}^\nu\big) = \big(\beta_\mu\beta^\mu\big)^2 = (U-T)^2\,. \tag{209}$$

To obtain the equations of motion of the string we need to use the force balance equation (112), which just takes the form

$$\overline{\nabla}_\mu\overline{T}^{\mu\nu} = \overline{f}^\nu\,, \tag{210}$$

for a string that is isolated, where $\overline{f}^\mu$ (which simply vanishes in the free case) is the force exerted by any background fields, such as the electromagnetic field and ambient (Higgs or other) fluid that are allowed for in the explicit expression (113). (The only difference if the string were not isolated, i.e. if it belonged to the boundary of one or more attached membranes, is that an additional contact force contribution $\check{f}^\mu$ of the form (112) would also be needed as well as the background contribution $\overline{f}^\mu$.) Using the Weingarten integrability condition (15) which is equivalent in the string case to the the projected Lie commutativity condition

$$\perp^\rho_\nu\big(\beta_+{}^\mu\overline{\nabla}_\mu\beta_-{}^\nu - \beta_-{}^\mu\overline{\nabla}_\mu\beta_+{}^\nu\big) = 0\,, \tag{211}$$

it can be seen from (207) that the extrinsic equations of motion governing the evolution of the string worldsheet will therefore [41] be expressible in characteristic form as

$$\perp^\rho_\nu\beta_\pm{}^\mu\overline{\nabla}_\mu\beta_\mp{}^\nu = \perp^\rho_\mu\overline{f}^\mu\,, \tag{212}$$

for either choice of the sign $\pm$, one version being obtainable from the other and vice versa by (211).

Except where the string state is locally degenerate in the sense that one of the bicharacteristic vectors $\beta_\pm{}^\mu$ is null, which by (210) will only occur where $T = U$ (as is everywhere the case for the Goto-Nambu model, for which both bicharacteristic vectors are null everywhere) it will be possible to fix their normalisation by requiring that they have the same magnitude, $\beta_{\pm\mu}\beta_\pm{}^\mu = -(T - U)$ and hence to define a corresponding pair of *unit bicharacteristic vectors*

$$u_\pm{}^\mu = \big(U - T\big)^{-1/2}\beta_\pm{}^\mu\,, \qquad u_{\pm\mu}u_\pm{}^\mu = -1\,. \tag{213}$$

5.2 Preferred orthonormal diad for nondegenerate state

In the generic case, $T < U$, for which the eigenvectors of the stress-energy tensor are non null, the corresponding unit vectors will constitute a preferred diad consisting of the unit timelike eigenvector u^μ introduced in (192), together with an orthogonal spacelike unit eigenvalue v^μ. These mutually orthogonal unit vectors will be related to the corresponding unit bicharacteristic vectors $u_+{}^\mu$ and $u_-{}^\mu$ (as given by (213) above) by the expressions

$$u^\mu = \frac{\sqrt{1 - c_{\rm E}{}^2}}{2}\big(u_+{}^\mu + u_-{}^\mu\big)\,, \qquad v^\mu = \frac{\sqrt{1 - c_{\rm E}{}^2}}{2c_{\rm E}}\big(u_+{}^\mu - u_-{}^\mu\big)\,. \tag{214}$$

In terms of this preferred diad, the antisymmetric unit surface element $\mathcal{E}^{\mu\nu}$ and the (first) fundamental tensor $\eta^{\mu\nu}$ of the string worldsheet will be given by

$$\mathcal{E}^{\mu\nu} = u^\mu v^\nu - v^\mu u^\nu\,, \qquad \eta^{\mu\nu} = -u^\mu u^\nu + v^\mu v^\nu\,, \tag{215}$$

and the standard form (193) for the surface stress momentum energy density tensor will reduce to the simple form

$$\overline{T}^{\mu\nu} = U u^\mu u^\nu - T v^\mu v^\nu\,. \tag{216}$$

Proceeding directly from this last form, the extrinsic dynamical equation (211), i.e. the surface orthogonal projection of (210), can be rewritten as

$$\perp^{\mu}_{\nu}\big(U\dot u^\nu - T v'^\nu\big) = \perp^{\mu}_{\nu}\overline{f}^\nu\,, \tag{217}$$

using the notation

$$\dot u^\mu = u^\nu\nabla_\nu u^\nu\,, \qquad v'^\mu = v^\nu\nabla_\nu v^\nu\,. \tag{218}$$

in terms of which the worldsheet curvature vector is expressible as

$$K^\mu = \perp^{\mu}_{\nu}(v'^\nu - \dot u^\nu)\,. \tag{219}$$

5.3 Duality in elastic string models

The formulae in Subsection 5.2 illustrate a special feature distinguishing string models from point particle models on one hand and from higher dimensional brane models on the other, namely the dual symmetry [36] that exists at a formal level between the timelike eigenvector u^μ and the associated eigenvalue U on one hand, and the the dynamically conjugate quantities that are the spacelike eigenvector and v^μ and the associated eigenvalue T. This feature is particularly informative in the barotropic or "perfectly elastic" case discussed at the end of the Section 4, i.e. in the case when the internal dynamics is controlled just by an equation of state giving the energy density U as a function of the tension T only, since in there will be a corresponding duality relation between the number density ν and the associated chemical potential μ as introduced in (197), which is equivalent to the mutually dual pair of differential defining relations

$$dU = \mu d\nu \ , \qquad dT = -\nu d\mu \, . \tag{220}$$

Defining the surface duals of the current vector ν^ρ and momentum covector μ_ρ given by (199) as

$$\star\nu_\rho = \mathcal{E}_{\rho\sigma}\nu^\sigma = \nu v_\rho \ , \qquad \star\mu^\rho = \mathcal{E}^{\rho\sigma}\mu_\sigma = \mu v^\rho \, , \tag{221}$$

one can use (201) to obtain the force balance equation for a barotropic string model in the self dual form

$$\mu_\rho \overline{\nabla}_\sigma \nu^\sigma + \star\nu_\rho \overline{\nabla}_\sigma \star\mu^\sigma + \perp^\sigma_\rho \big(U\dot{u}^\sigma - Tv'^\sigma \big) = \overline{f}_\rho \, , \tag{222}$$

whose orthogonally projected part gives back the self dual extrinsic force balance equation (217), while its tangentially projected part gives a mutually dual pair of surface current source equations of the form

$$(U-T)\overline{\nabla}_\sigma \nu^\sigma = -\nu^\rho \overline{f}_\rho \, , \qquad (U-T)\overline{\nabla}_\sigma \star\mu^\sigma = \star\mu^\rho \overline{f}_\rho \, . \tag{223}$$

It can be seen from this that, like the surface number current density ν^σ, the dual current $\star\mu^\sigma$ will also be conserved in a free barotropic string, i.e. when the external force $\overline{f}_\rho$ is absent.

It is useful [36] to introduce a dimensionless state function called the "characteristic potential" ϑ say that is constructed in such a way as to be self dual according to the differential relation

$$d\vartheta = \sqrt{\frac{d\mu \, d\nu}{\mu\nu}} \, . \tag{224}$$

This function is convenient for the purpose of writing the internal equations of motion obtained by tangential projection of (210) in a characteristic form that will be the analogue of the characteristic version (212) of the extrinsic equations of motion obtained by the orthogonal projection of (210). The longitudinal bicharacteristic unit vectors $\ell_\pm{}^\mu$ say (the analogues for the internal sound type

waves of the extrinsic bicharacteristic unit vectors ${u_\pm}^\mu$ introduced above) are defineable by

$$ {\ell_\pm}^\mu = \big(1 - {c_{\rm L}}^2\big)^{-1/2}\big(u^\mu \pm c_{\rm L}\, v^\mu\big)\ , \qquad {\ell_\pm}_\mu {\ell_\pm}^\mu = -1\,, \tag{225} $$

where $c_{\rm L}$ is the longitudinal characteristic speed as given by (204). In terms of these, the tangentially projected part of the force balance equation (210) is convertible [36] to the form of a pair of scalar equations having the characteristic form

$$ (U-T){\ell_\pm}^\rho\big(\overline{\nabla}_\rho \vartheta \mp v_\nu \overline{\nabla}_\rho u^\nu\big) = \pm\mathcal{E}_{\mu\nu}\,{\ell_\mp}^\mu\,\overline{f}^\nu\,, \tag{226} $$

from which it is directly apparent that, as asserted in the Section 4, the state function $c_{\rm L}$, as given by (204), is indeed the characteristic speed of longitudinal "woggle" propagation relative to the preferred frame specified by the timelike eigenvector u^μ. When rewritten in terms of the unit bicharacteristic vectors ${u_\pm}^\mu$ that are the extrinsic analogues of ${\ell_\pm}^\mu$, the corresponding extrinsic equations (212) take the form

$$ (U-T)\perp^{\!\mu}_{\!\nu}{u_\pm}^\rho\overline{\nabla}_\rho\,{u_\mp}^\nu = \perp^{\!\mu}_{\!\nu}\overline{f}^\nu\,, \tag{227} $$

(It is to be noted that the corresponding equation (32) at the end of the original discussion [36] of the characteristic forms of the equations of string motion contains a transcription error whereby a proportionality factor that should have been $\sqrt{(U-T)T}$ was replaced by its non-relativistic – i.e. low tension – limit, namely $\sqrt{UT}$.)

5.4 *The Hookean prototype example of an elastic string model

What essentially distinguishes different physical kinds of "perfectly elastic", i.e. barotropic, string model from one another is the form of the equation of state, which depends on the microscopic internal details of the vacuum defect (or other structure) that the string model is supposed to represent. In a cosmological context, the most important special case, which will be discussed in detail later on, is given by the constant product formula (205), but for terrestrial applications the most useful kind of equation of state for general purposes is the one named after Robert Hooke, who is recogniseably the principal founder of string theory as a branch of physics in the modern sense of the word. (As for superstring theory, it remains to be seen whether it will become established as a branch of physics or just as a topic of mathematics.) Hooke's famous law to the effect that the tension, T is proportional to the extension, i.e. the change in $1/\nu$ in the notation used here, provides a very good approximation in the low tension limit that is exemplified by applications such as that of an ordinary violin string. This law is expressible within the present scheme as

$$ U(Y+T) = U_0 Y + \tfrac{1}{2}T^2\,, \tag{228} $$

where U_0 is a constant interpretable as the rest frame energy in the fully relaxed (zero tension) limit, and Y is another constant interpretable as what is known (after another pioneer worker on the subject) as Young's modulus.

5.5 General elastic string models

A more general category of non-barotropic string models is of course required, not only in such terrestrial applications as (above ground) electric power transmission cables (the subject of my first research project as an undergraduate working working for commercial industry during a vacation), but also for many cosmologically relevant cases that might be envisaged (such as that of a "warm superconducting string" [94]) in which several independent currents are present. Nevertheless most of the cosmological string dynamical studies that have been carried out so far have been restricted (not just for simplicity, but also as a very good approximation in wide range of circumstances) to models of the the "perfectly elastic" barotropic type, and more specifically to the category (including the Hookean and Goto Nambu examples as opposite extreme special cases) of string models describable by a Lagrangian of the form

$$\overline{\mathcal{L}} = \Lambda + \overline{j}^{\mu} A_{\mu} + \tfrac{1}{2}\overline{w}^{\mu\nu} B_{\mu\nu}\,, \qquad (229)$$

in which the only independent field variable on the world sheet can be taken to be a stream function ψ, which enters only via the corresponding identically conserved number density current

$$c^{\mu} = \mathcal{E}^{\mu\nu} p_{\nu}\,, \qquad\qquad p_{\nu} = \overline{\nabla}_{\nu}\psi\,, \qquad (230)$$

whose squared magnitude,

$$\chi = c_{\mu}c^{\mu} = -p^{\mu}p_{\nu} = -\gamma^{ij}\psi_{i}\psi_{j}\,, \qquad (231)$$

is the only argument in a (generically non-linear) *master function*,

$$\Lambda = \Lambda\{\chi\}\,. \qquad (232)$$

(It is to be noted that there was a transcription error whereby the factor $^1/_2$ in the final term of (229) was left out in the preceeding version of these notes [1].) In order to satisfy the vorticity flux conservation requirement (as derived in Section 2 from the condition of invariance with respect to gauge changes in the background Kalb-Ramond field $B_{\mu\nu}$) one must have

$$\overline{w}^{\mu\nu} = \overline{h}\mathcal{E}^{\mu\nu}\,, \qquad (233)$$

for some fixed (quantised) circulation value $\overline{h}$ say – a sort of helicity constant associated with the string. In the simplest "local" string models [36], [37] this quantity $\overline{h}$ can simply be taken to vanish, while in the in the simplest "global" models [32], [34], [35] it will be directly identifiable with Planck's constant, i.e. one will have $\overline{h} = 2\pi\hbar$. (In more general cases it might be an integral multiple of

this.) The electromagnetic current density is specified in terms of the gradient of the stream function in the form

$$\overline{j}^{\mu} = q c^{\mu} \,, \tag{234}$$

where q is another fixed parameter which is interpretable as the electromagnetic charge coupling constant per idealised particle of the current, which means that its value will depend on the convention used to fix the normalisation of ψ or equivalently of c^{μ}, a question that will be taken up in the discussion following equation (247) below.

Previous studies of cosmic strings have mostly been restricted to cases involving allowance for a Kalb-Ramond coupling (which is relevant for vortex defects of "global type" [32], [33], [34], [95], [96] in the absence of an internal current, or alternatively to cases (such as string models of the kind [36], [9] needed to describe local superconducting vortices arising from the Witten [8]) mechanism involving allowance for an internal current in the absence of Kalb-Ramond coupling. However there is no particular difficulty in envisaging the presence of both kinds of coupling simultaneously as is done here. A new result obtained thereby (in 6) is that for stationary and other symmetric configurations, the effect of the coupling to the Kalb Ramond background is expressible as that of a fictitious extra electromagnetic field contribution.

It is the specific form of the master function that determines the relevant equation of state – or if its range of validity is sufficiently extensive, the pair of equations of state, corresponding to the qualitatively different regimes distinguished by the positivity or negativity of χ – by which the dynamics of the model is governed. The derivative of the master function provides a quantity

$$\mathcal{K} = 2\frac{d\Lambda}{d\chi} \,, \tag{235}$$

in terms of which the surface stress momentum energy density tensor defined by (90) will be given by

$$\overline{T}^{\mu\nu} = \mathcal{K} p^{\mu} p^{\nu} + \Lambda \eta^{\mu\nu} \,. \tag{236}$$

This only depends on the internal field gradient and on the imbedding and the background geometry: it involves neither the background electromagnetic and Kalb Ramond fields nor their respective coupling constants q and $\overline{h}$. The role of the latter is just to determine the background force density, which will be given, according to the general formula (113), by

$$\overline{f}_{\rho} = q F_{\rho\mu} c^{\mu} + \frac{\overline{h}}{2} N_{\rho\mu\nu} \mathcal{E}^{\mu\nu} \,. \tag{237}$$

Unlike the force density, the stress momentum energy density tensor, as given by (236), is formally invariant under a duality transformation [5], [36] whereby the variational momentum covector p_{μ}, the current c^{μ}, and the master function,

Λ itself, are interchanged with their duals, as denoted by a tilde, which are definable by

$$\tilde{c}^\mu = \mathcal{K} p^\mu \,, \qquad \tilde{p}_\mu = \mathcal{K} c_\mu \,, \qquad \tilde{\chi} = \tilde{c}_\mu \tilde{c}^\mu = -\tilde{p}^\nu \tilde{p}_\nu \,, \tag{238}$$

and by the relation

$$\chi \mathcal{K} = \Lambda - \tilde{\Lambda} = -\tilde{\chi} \tilde{\mathcal{K}} \,, \tag{239}$$

with

$$\tilde{\mathcal{K}} = 2 \frac{d\tilde{\Lambda}}{d\tilde{\chi}} = \mathcal{K}^{-1} \,, \qquad \tilde{\chi} = -\mathcal{K}^2 \chi \,. \tag{240}$$

In terms of these quantities the stress momentum energy density tensor can be rewritten in either of the equivalent mutually dual canonical forms

$$\tilde{c}^\mu p_\nu + \Lambda \eta^\mu{}_\nu = \overline{T}^\mu{}_\nu = c^\mu \tilde{p}_\nu + \tilde{\Lambda} \eta^\mu{}_\nu \,. \tag{241}$$

Substitution of these respective expressions in (210) then gives the force balance equation in the equivalent mutually dual forms

$$p_\rho \overline{\nabla}_\mu \tilde{c}^\mu + 2\tilde{c}^\mu \overline{\nabla}_{[\mu} p_{\rho]} + \Lambda K_\rho = \overline{f}_\rho = \tilde{p}_\rho \overline{\nabla}_\mu c^\mu + 2 c^\mu \overline{\nabla}_{[\mu} \tilde{p}_{\rho]} + \tilde{\Lambda} K_\rho \,. \tag{242}$$

The independent internal equations of motion of the system are obtainable by contraction with the independent mutually orthogonal tangent vectors p^ρ and $\tilde{p}^\rho$, the latter giving what is just a kinematic identity, since, by construction according to (237) one automatically has

$$\tilde{p}^\rho \overline{f}_\rho = 0 \tag{243}$$

and hence

$$\overline{\nabla}_\mu c^\mu = 0 \quad \Leftrightarrow \quad \mathcal{E}^{\rho\mu} \overline{\nabla}_\mu p_\rho \ = 0. \tag{244}$$

The only part of the internal equations of motion that is properly dynamical (from the point of view of the variation principle with respect to Λ) is thus just the part got by contraction with p^ρ: since (237) gives

$$p^\rho \overline{f}_\rho = q \chi \mathcal{E}^{\mu\rho} F_{\mu\rho} \,. \tag{245}$$

one obtains the equivalent alternative forms

$$\overline{\nabla}_\mu \tilde{c}^\mu = q \mathcal{E}^{\mu\rho} F_{\rho\mu} \quad \Leftrightarrow \quad \mathcal{E}^{\mu\rho} \big(2 \overline{\nabla}_{[\mu} \tilde{p}_{\rho]} + q F_{\mu\rho} \big) = 0 \,. \tag{246}$$

It is to be noted that unlike the electromagnetic force, the Joukowski force (interpretable as a manifestation of the Magnus effect) arising from the Kalb-Ramond coupling in (237) will always act orthogonally to the world sheet and thus does not affect the internal dynamics of the string.

5.6 Standard normalisation of current and charge

The preceeding equation (246) can be construed as the (Poincaré type) integrability condition for the local existence on the string worldsheet of a scalar φ in terms of which the dual momentum will simply be given as the tangentially projected gauge covariant derivative:

$$\tilde{p}_\rho = \sharp \overline{D}_\rho \varphi \ , \qquad \overline{D}_\rho = \eta^\mu{}_\rho D_\mu \ , \qquad D_\mu = \nabla_\mu - eA_\mu \ , \tag{247}$$

where $\sharp$ is a normalisation constant and e is a new, implicitly more fundamental, charge coupling constant in terms of which the original charge per unit idealised particle of the current is given by

$$q = \sharp e \, . \tag{248}$$

The scalar φ can be used as the primary independent variable, instead of ψ, in an alternative (dynamically conjugate) variational formulation [5], [36] using a Lagrangian constructed from $\tilde{\Lambda}$ instead of Λ.

By adjusting the scale of calibration used for the measurement of either the current c^μ or the field φ it would of course always be possible, for purposes of mathematical convenience to simply set the dimensionless scale factor $\sharp$ to unity. However other choices may be more useful for purposes of physical interpretation. In particular, for the kind of string model [9] that provides a macroscopic descrition of Witten type syperconducting vortices, the scaling of φ can most naturally be chosen in such a way as to allow this field to be interpreted as the phase of the complex scalar field characterising the underlying bosonic condensate, so that the latter will be expressible in the form $\Phi = |\Phi| \mathrm{e}^{i\varphi}$. On the other hand it has been found mathematically convenient in the same context to fix the normalisation of the current c^μ in a standard manner by requiring that the coefficient $\mathcal{K}$ introduced in (235) should tend to unity in the null current limit, i.e. as χ tends to zero. The condition for this standard current normalisation to be consistent with the natural normalisation of φ is expressible as the requirement that the ratio $\sharp$ (interpretable as the number of fundamental particle units per idealised particle unit of the current c^μ) should just be a square root of the model dependent constant κ_0 that was used in previous work [9], i.e. $\sharp^2 = \kappa_0$. The corresponding fundamental particle current vector z^μ say will thus be given by

$$z^\mu = \sharp c^\mu \ , \tag{249}$$

so that if the particles are charged, with fundamental charge coupling constant e, the corresponding electromagnetic surface current vector (234) will be given simply by

$$\overline{j}^\mu = e z^\mu \, . \tag{250}$$

If its range of definition is sufficiently extended, the master function Λ will determine not just one but a pair of distinct equations of state, one applying to the "magnetic" regime where the current c^μ is spacelike so that χ is positive, and the other applying to the "electric" regime where the current c^μ is timelike

so that the χ is negative. In either of these distinct regimes – though not on the critical intermediate "null state" locus where the current is lightlike so that χ vanishes – the mutually dual *canonical* forms (241) of the stress momentum energy density tensor are replaceable by the equivalent more manifestly symmetric *standard* form

$$\overline{T}^{\mu\nu} = \frac{\Lambda}{\chi}\, c^\mu c^\nu + \frac{\tilde\Lambda}{\tilde\chi}\, \tilde c^\mu \tilde c^\nu \,. \tag{251}$$

By comparison with the corresponding expression (216) it can thus be seen that the state functions introduced in the variational function that has just been specified will be phyiscally interpretable in terms of the corresponding energy density U and tension T, and also of the number density ν and effective mass μ associated with the corresponding physically normalised particle flux (meaning that for which the charge per particle is e) will be given parametrically in the "electric" regime where c^μ is timelike, by

$$\Lambda = -U\,, \qquad \tilde\Lambda = -T\,, \qquad \sharp^2\chi = -\nu^2\,, \qquad \tilde\chi = \sharp^2\mu^2\,, \qquad \mathcal{K} = \sharp^2\frac{\mu}{\nu}\,, \tag{252}$$

with

$$\sharp c^\rho = \nu u^\rho\,, \qquad \tilde c^\rho = \sharp\mu v^\rho\,, \qquad \sharp p_\rho = \nu v_\rho\,, \qquad \tilde p_\rho = \sharp\mu u_\rho\,. \tag{253}$$

(For the corresponding mathematically normalised flux, whose idealised particles are to be considered as each containing $\sharp$ of the fundamental particles – so that, in accordance with (248) the corresponding charge per particle is $\sharp e$ – the corresponding effective mass would be $\sharp\mu$ and the corresponding number density would be $\nu/\sharp$.) The analogous formulae for the "magnetic" regime where c^μ is spacelike (the first possibility to be considered in the early work on the Witten mechanism [8]), will be given by

$$\Lambda = -T\,, \qquad \tilde\Lambda = -U\,, \qquad \sharp^2\chi = \mu^2\,, \qquad \tilde\chi = -\sharp^2\nu^2\,, \qquad \mathcal{K} = \sharp^2\frac{\nu}{\mu}\,, \tag{254}$$

with

$$\sharp c^\rho = \mu v^\rho\,, \qquad \tilde c^\rho = \sharp\nu u^\rho\,, \qquad \sharp p_\rho = \mu u_\rho\,, \qquad \tilde p_\rho = \sharp\nu v_\rho\,. \tag{255}$$

Whether or not it is charged, and whatever the local (spacelike, timelike, or null) character of the current may be, the conservation law (236) implies that its integral round any *closed* string loop will give a corresponding conserved particle quantum number

$$Z = \oint dx^\mu \mathcal{E}_{\mu\nu} z^\nu\,. \tag{256}$$

For such a loop there will also be a second independent topologically conserved phase winding quantum number, N say, defined by

$$2\pi N = \oint d\varphi\,. \tag{257}$$

It can be seen from (247) that this number will be given by

$$2\pi\sharp N = q\Phi + \oint dx^\mu\, \tilde p_\mu\,, \tag{258}$$

where Φ is the magnetic flux integral,

$$\Phi = \oint dx^\mu A_\mu \,. \tag{259}$$

It is to be noted that in the electromagnetically charged coupled case, i.e. when the relevant fundamental particle coupling constant e has a non zero value (whose approximate value will be given in fundamental units by $1/\sqrt{137}$ or a multiple thereof) the loop will have a conserved total electric charge, Q say, given by

$$Q = eZ = \oint dx^\mu \mathcal{E}_{\mu\nu} \bar{\jmath}^\nu \,. \tag{260}$$

5.7 Analytic string model for Witten vortex

In realistic conducting string models one can not expect the appropriate master function $\Lambda\{\chi\}$ to be given exactly by an explicit analytic formula, but only by the output of a detailed numerical computation of the internal structure of the underlying vacuum defect [97], [75], [76], [77]. Nevertheless it has recently been found [9] that a remarkably good approximated description of the results of such computations for Witten's original "superconducting" bosonic vortex model can be provided by analytic expressions of a fairly simple kind, involving only two model dependent mass parameters. The one that is presumed to be the larger is the usual Kibble mass m say, whose square determines the energy and tension in the null current limit, and whose value is expected to be of the order of the mass of the Higgs field responsible for the spontaneous symmetry breaking that produces the vortex defect. The other mass parameter, m_* say, is expected to be of the order of the considerably smaller mass scale that is presumed to characterise the independent current carrier field in the relevant version of Witten's bosonic field model [8]. In all the cases that have been investigated in detail [75], [76], [77] it has been found that to obtain a reasonably accurate representation throughout the ("magnetic") regime for which $\tilde\chi$ is negative, and for moderate positive values as well, it suffices [9] to use the purely polynomial formula

$$\tilde\Lambda = -m^2 + \frac{\tilde\chi}{2}\Big(1 - \frac{\tilde\chi}{2{m_*}^2}\Big)^{-1} , \tag{261}$$

which is applicable within the range

$$-\frac{1}{3} < \frac{\tilde\chi}{2{m_*}^2} < 1 - \frac{{m_*}^2}{m^2 + {m_*}^2} \,, \tag{262}$$

which is limited below by the condition that ${c_{\rm L}}^2$ should be positive and limited above by the condition that ${c_{\rm E}}^2$ should be positive. For values near the upper end of this range (in the "electric" regime) the formula (261) does however, becomes inadequate, but to obtain a very satisfactory description throughout the positive range of $\tilde\chi$, it suffices to replace it by the still very simple though

non-polynomial formula

$$\tilde{\Lambda} = -m^2 - \frac{m_*{}^2}{2} \ln\Big\{1 - \frac{\tilde{\chi}}{m_*{}^2}\Big\}\,, \tag{263}$$

for which the corresponding range of applicability is given by

$$-1 < \frac{\tilde{\chi}}{m_*{}^2} < 1 - \mathrm{e}^{-2m^2/m_*{}^2}\,. \tag{264}$$

Since the underlying field theoretical model proposed by Witten [8] provides what can at best be only a highly simplified description of the mechanisms that would be involved in any real vacuum vortex, there is no point in investing too much effort in describing it with unduly high precision. To provide just the moderate degree of accuracy that is physically reasonable, the single formula (263) is all that is needed to cover the entire admissible range: it is more accurate than it need be for this purpose in the positive range of $\tilde{\chi}$; it agrees perfectly with the preceeding formula (261) in the null limit where $\tilde{\chi}$ tends to zero; and finally even in the negative ("magnetic") range, where (261) is quantitatively more accurate, the newer formula (263) still provides a description that is in qualitatively adequate agreement with that of its predecessor. In the ("electric") range $\tilde{\chi} > 0$ where the formula (263) is most accurate, the equation of state relating the energy U to the tension T will take the form

$$U = T + m_*{}^2\big(\mathrm{e}^{2(m^2 - T)/m_*{}^2} - 1\big)\,, \tag{265}$$

while in the ("magnetic") range the corresponding relation is given by

$$T = U - m_*{}^2\big(1 - \mathrm{e}^{-2(U - m^2)/m_*{}^2}\big)\,. \tag{266}$$

It is evident that in the absence of background forces, knowledge of the two parameters m and m_* is fully sufficient to characterise the dynamics of such a string model – indeed it is only the ratio m/m_* that actually matters for this purpose. However one will also have to evaluate the relevant scaling factor $\sharp = \sqrt{\kappa_0}$ (which, like m and m_* will depend on the parameters characterising the underlying bosonic field theory) if one wishes to be able to allow for the effect of any background electromagnetic field that may be present, since for this purpose it is necessary to know the value of the mathematically normalised coupling constant q. The latter will be directly obtainable from the value of $\sharp$ (which in typical cases may be expected to be of the order of unity) via (248), on the assumption that the relevant value of the physical charge coupling constant e characterising the boson condensate is just the usual electronic charge (as given in Planck units by $e^2 \approx 1/137$) or a simple integral multiple thereof.

5.8 *The self-dual transonic string model

Although the formulae [9] in the preceeding section provide the simplest description that one could reasonably desire for the purpose of describing realistic

superconducting strings arising from the Witten mechanism, it is instructive to study an even simpler model that originally turned up in a rather different physical context of a more artificial nature, namely that of a Kaluza Klein type projection of a simple Goto-Nambu model in an extended background with an extra space dimension in the manner first suggested by Nielsen [85]. It can easily be shown [5] that this gives rise to a particularly elegant illustration of the above formalism in which the relevant master function has the rather unusual though by no means unique property of being self dual in the sense that (for an appropriate choice of the scaling of the stream function ψ) the dependence of the master function Λ on its argument χ is exactly the same as that of the corresponding function $\tilde{\Lambda}$ on $\tilde{\chi}$, their form in this case being is given in terms of a constant mass parameter m simply by

$$\Lambda = m\sqrt{m^2-\chi}\ , \qquad \tilde{\Lambda} = m\sqrt{m^2-\tilde{\chi}}\,. \tag{267}$$

This self duality property means that the equation of state relationship between U and T will the *same* form in the magnetic regime $\chi > 0 > \tilde{\chi}$ as in the electric regime $\tilde{\chi} > 0 > \chi$. What is quite unique about the equation of state obtained for particular self dual string model is that it will have the non-dispersive – i.e. permanently transonic – "constant product" form (205), which can be seen [89] to be expressible parametrically in terms of a dimensionless self dual state function ϑ of the kind introduced in (224) by

$$U = m^2\sqrt{1+\frac{\nu^2}{m^2}} = m^2\coth\vartheta\ , \qquad T = m^2\sqrt{1-\frac{\mu^2}{m^2}} = m^2\tanh\vartheta\,, \tag{268}$$

which gives

$$c_{\rm E} = c_{\rm L} = \tanh\vartheta\ , \tag{269}$$

so that by (205) and (204), the extrinsic ("wiggle") and longitudinal ("woggle" or "jiggle") bicharacteristic vectors will coincide, having the form

$$\ell_\pm{}^\mu = u_\pm{}^\mu = \cosh\vartheta\, u^\mu \pm \sinh\vartheta\, v^\mu\ . \tag{270}$$

The revelation [5] of the special transonicity property of the model characterised by (267) was sufficient to invalidate the over hasty claim [86] that Nielsen's elegant artifice [85] effectively represents the outcome in the "pure" string limit (in which effects of finite vortex thickness are neglected) of the Witten mechanism [8]. Long before the derivation [9] of the explicit formula (263) it was already known that in order to provide a satisfactory representation a Witten type vacuum vortex it would be necessary to use a model [97] of generically dispersive type [36]. More specifically, it was known from the work of Peter [75], [76], [77] that such a model would need to be characterised typically by supersonicity, $c_{\rm E} > c_{\rm L}$ for spacelike and moderate timelike values of the current provided the ratio of the carrier mass to the Higgs mass is not too large. Now that the model (263) is available, it can be verified directly that it will indeed exhibit supersonicity for all spacelike values and at least for small timelike values of the current not just when the ratio ${m_*}^2/m^2$ is small compared unity but

so long as it does not exceed the implausibly large value 2. On the other hand it can be seen that such model will always become subsonic for sufficiently large timelike current values (near the relaxation limit in the "electric" regime).

It is to be remarked that despite its inability to reproduce these features, the use of the mathematically convenient transonic model (267), as advocated by Nielsen and Olesen [86], for the purpose of describing the dynamics of a realistic superconducting string model was nevertheless a considerable improvement, not just on the use [73] for this purpose of an unmodified Goto-Nambu model, but also on the more commonly used description provided by the naively linearised model [68], [69], [70], [71], [72], [74] with equation of state given by constancy of the trace (208) which, by (194) and (204), is evidently characterised by permanent subsonicity, $c_{\rm E} < c_{\rm L} = 1$.

The intrinsic equations of motion (223) for the transonic model can be usefully be recombined [40], [94] as an equivalent pair of divergence relations that are expressible in terms of the bicharacteristic vectors (270), in the form

$$\overline{\nabla}_\rho\Big((U-T)\ell_\pm{}^\rho\Big) = -\ell_\mp{}^\rho\overline{f}_\rho\,, \tag{271}$$

which shows that the "left" and "right" moving "bicharacteristic currents", $(U-T)\ell_+{}^\rho$ and $(U-T)\ell_-{}^\rho$ will each be conserved separately in the free case, i.e. when $\overline{f}_\rho$ vanishes. As an alternative presentation of the tangential force balance equations for this permanently transonic model, a little algebra suffices to show that the intrinsic equation of motion (226) can be rewritten for this case, in a form more closely analogous to that of the corresponding extrinsic equation of motion (227), as the pair of equations

$$(U-T)\eta^\mu{}_\nu\,\ell_\pm{}^\rho\overline{\nabla}_\rho\ell_\mp{}^\nu = \big(\eta^\mu{}_\nu + \ell_\mp{}^\mu\ell_{\mp\nu}\big)\overline{f}^\nu\,. \tag{272}$$

This tangentially projected part of the dynamic equations can now be recombined [89] with its orthogonally projected analogue (5.22), so as to give the *complete* set of force balance equations for the non dispersive permanently transcharacteristic string model (4.21) as the extremely useful pair of bicharacteristic propagation equations

$$(U-T)\ell_\pm{}^\rho\overline{\nabla}_\rho\ell_\mp{}^\mu = \big(g^{\mu\nu} + \ell_\mp{}^\mu\ell_\mp{}^\nu\big)\overline{f}_\nu\,. \tag{273}$$

5.9 *Integrability and application of the transonic string model

It is evident that the bicharacteristic formulation (273) of the equations of motion for the transonic string model will take a particularly simple form when translated into terms of the corresponding characteristic coordinates $\sigma^\pm$ on the worldsheet, where the latter are defined by taking the "right moving" coordinate σ^+ to be constant allong "left moving" characteristic curves, and taking the "left moving" coordinate σ^- to be constant along "right moving" characteristic curves, with the convention that the correspondingly parametrised bicharacteristic tangent vectors $\ell^\mu_\pm \equiv \partial x^\mu/\partial\sigma^\pm$ should be future directed.

In the case of a *free* motion (i.e. when the force term on the right of (273) vanishes) in a *flat* background, this simplification can be used [89] to obtain the complete solution of the dynamical equations in a very simple explicit form. The way this works is that with respect to Minkowski coordinates in such a background (273) will reduces to the simple form

$$\partial \ell_{\pm}^{\nu} / \partial \sigma_{\mp} = 0 \ , \tag{274}$$

whose general solution is given in terms of a pair of generating curves $x_{\pm}^{\mu}\{\sigma\}$ as a sum of single variable functions by the ansatz

$$x^{\mu} = x_{+}^{\mu}\{\sigma^{+}\} + x_{-}^{\mu}\{\sigma^{-}\} \ , \tag{275}$$

which gives $\ell_{\pm}^{\mu} = \dot{x}_{\pm}^{\mu}$ using the dot here to denote the ordinary derivatives of the single variable functions with respect to the corresponding characteristic variables. The solution (275) generalises a result that is well known for more familiar but degenerate Goto Nambu case, in which the tangents to the generating curves are required to be null, $\ell_{\pm}^{\mu}\ell_{\pm\mu} = 0$ (so that with the usual normalisation their space projections lie on what is known as the Kibble Turok sphere [98]). The only restriction in the non - degenerate case is that they should be non-spacelike and future directed (the corresponding projections thus lying anywhere in the interior, not just on the surface, of a Kibble Turok sphere) the unit normalisation condition (225) being imposable as an option, not an obligation, by choosing the parameter σ to measure *proper* time along each separate generating curve.

This special property of being explicitly integrable by an ansatz of the same form (275) as has long been familiar for the Goto Nambu case can immediately be used to provide a new direct demonstration of the validity of the claim I originally made several years ago [89] to the effect that the non-dispersive constant product model (205) is not just a mathematical curiosity, but that it is usefully applicable for the purpose of providing a realistic description of the average motion of a “wiggly” Goto Nambu string, effectively extending to the tension T the concept of “renormalisation” that had previously been introduced for the energy density U by Allen and Shellard [99], [100].

The new justification [92] summarised here is needed because my original argument [89] was merely of a qualitative heuristic nature, while Vilenkin’s mathematical confirmation [90] was based on indirect energetic considerations, and was legitimately called into question [91] on the grounds that it did not cover quite the most general class of “wiggles” that can be envisaged. Such an objection is effectively bypassed by the alternative approach presented here, but there would still be a paradox if allowance for more general “wiggles” really did produce what was alleged: thus although it was based on manifestly muddled reasonning and misinterpretation of preceding work, the purported demonstration [91] of higher order “deviations” from the constant product form (205) for the effective equation of state gave rise to a controversy that was not definitively resolved until a genuinely valid extension of Vilenkin’s method [90] was finally produced by Martin [93], whose conclusion was that the constant product

equation of state (205) will after all provide an account that remains accurate for *arbitrary* "wiggle" perturbations of a Goto Nambu string (subject only to the restriction that their amplitudes should not be so large as to bring about a significant rate of self intersection).

As an alternative to the energy analysis originated by Vilenkin and completed by Martin, the more direct justification [92] for the use of the elastic string model characterised by (267) as a model for the large scale averaged behaviour of a Goto-Nambu string is as follows. The argument is simply based on the observation that such a model implicitly underlies the diamond lattice discretisation that, since its original introduction by Smith and Vilenkin [101], has been commonly employed by numerical simulators [102] as a very convenient approximation scheme – of in principle unlimited accuracy – for the representation of a Goto-Nambu string worldsheet in a flat background. As a way of replacing the exact continouous description by a discrete representation such as is necessary for numerical computation, the idea of the Smith Vilenkin method is simply to work with a pair of discrete sets of sampling points $x_{\pm\mathrm{r}}^{\ \mu} = x_{\pm}^{\mu}\{\sigma_{\mathrm{r}}\}$ determined by a corresponding discrete set of parameter values σ_{r} on the generating curves of the exact representation (275). This provides a "diamond lattice" of sample points given (for integral values of r and s) by

$$x_{\mathrm{rs}}^{\nu} = x_{+\mathrm{r}}^{\ \mu} + x_{-\mathrm{s}}^{\ \mu}\,, \tag{276}$$

that will automatically lie exactly on the "wiggly" Goto Nambu worldsheet (275), which is thus represented to any desired accuracy by choosing a sufficiently dense set of sampling parameter values σ_{r} on the separate "wiggly" null generating curves $x_{\pm}^{\ \mu}\{\sigma\}$. It is evident that the chosen set of sample points $x_{\pm\mathrm{r}}^{\ \mu} = x_{\pm}^{\mu}\{\sigma_{\mathrm{r}}\}$ on the separate "wiggly" null generators can also be considered to be sample points on a pair of *smoothed out*, and thus no longer null but *timelike*, interpolating curves that, according to the result [89] demonstrated above, can be interpreted according to (275) as generating a corresponding solution of the equations of motion for an elastic string model of the kind governed by (205). The not so "wiggly" elastic string worldsheet constructed by this smoothing operation will obviously be an even better approximation to the exact "wiggly" Goto Nambu worldheet than the original Smith Vilenkin lattice representation, which itself could already be made as accurate as desired by choosing a sufficiently high sampling resolution. No matter how far it is extrapolated to the future, the smoothed elastic string worldsheet generated according to (274) can never deviate significantly from the underlying "wiggly" Goto-Nambu worldsheet it is designed to represent because the exact worldsheet and the smoothed interpolation will always coincide precisely at each point of their shared Smith Vilenkin lattice (276). This highly satisfactory feature of providing a potentially unlimited accuracy could not be improved but would only be spoiled by any "deviation" from the originally proposed [89] form (205) for the effective equation of state.

After thus conclusively establishing that the permanently transonic elastic string model characterised by the simple constant product equation of state

(205) (without any higher order corrections) provides an optimum description of the effect of microscopic wiggles in an underlying Goto Nambu model so long as self intersections remain unimportant (as was assumed in all the discussions [89], [90], [91], [92], [93] cited above), it remains to be emphasised that the neglect of such intersections will not be justified when the effective temperature [89], [94] of the wiggles is too high (as will presumably be the case [40] during a transient period immediately following the string - forming phase transition). The result of such intersections will be the formation of microscopic loops, of which some will subsequently be reconnected, but of which a certain fraction will escape. Estimation of the dissipative cooling force density that would be needed to allow for such losses remains a problem for future work.

6 Symmetric configurations including rings and their vorton equilibrium states

6.1 Energy-momentum flux conservation laws

Whenever the background space time metric is invariant under the action of a (stationarity, axisymmetry, or other) continuous invariance group generated by a solution ℓ^μ Killing of the equation (118), i.e. in the notation of (106)

$$\vec{\ell}\pounds g_{\mu\nu} = 0\,, \tag{277}$$

then any string that is isolated (i.e. not part of the boundary of an attached membrane) will have a corresponding *momentum* current (interpretable, depending on the kind of symmetry involved, as representing a flux of energy, angular momentum, or whatever) given by

$$\overline{P}^\mu = \overline{T}^\mu{}_\nu \ell^\nu\,. \tag{278}$$

In accordance with (120), this will satisfy a source equation of the form

$$\overline{\nabla}_\mu \overline{P}^\mu = \overline{f}_\mu \ell^\mu\,, \tag{279}$$

which means that the corresponding flux would be strictly conserved when the string were not just isolated but *free*, i.e. if the background force $\overline{f}_\mu$ were zero. When the string is subject to a background force of the Lorentz-Joukowsky form (238) that arises from background electromagnetic and Kalb-Ramond fields, then provided these background fields are also invariant under the symmetry group action generated by ℓ^μ, i.e. in the notation of (106)

$$\vec{\ell}\pounds A_\mu = 0\,, \qquad \vec{\ell}\pounds B_{\mu\nu} = 0\,, \tag{280}$$

it can be seen to follow that, although the physically well defined surface current $\overline{P}^\mu$ will no longer be conserved by itself, it still forms part of a gauge dependent generalisation, $\overline{\mathcal{P}}^\mu$ say, that is strictly conserved,

$$\overline{\nabla}_\mu \mathcal{P}^\mu = 0\,, \tag{281}$$

and that is given, in terms of the gauge dependent generalisation

$$\overline{\mathcal{T}}^{\mu}{}_{\nu} = \overline{T}^{\mu}{}_{\nu} + \overline{j}^{\mu} A_{\mu} + \overline{w}^{\mu\rho} B_{\nu\rho} \tag{282}$$

of the surface stress momentum energy density tensor, by

$$\overline{\mathcal{P}}^{\mu} = \overline{\mathcal{T}}^{\mu}{}_{\nu}\ell^{\nu} = \overline{P}^{\mu} + \big(qc^{\mu} A_{\nu} + \overline{h}\mathcal{E}^{\mu\rho} B_{\nu\rho}\big)\ell^{\mu}\,. \tag{283}$$

6.2 Bernoulli constants for a symmetric string configuration

Let us now restrict attention to cases in which string configuration itself shares the background symmetry under consideration. This category includes the astrophysically interesting case of circular configurations in an axisymmetric background – which is of particular interest in the context of vorton formation in a flat background – and of course it also includes many familiar terrestrial examples of static string configurations in a stationary background – one of the most obvious being that of the overhead telephone and power cables that festoon our environment. Any string configuration that is symmetric in this sense will be characterised by the condition

$$\perp^{\mu}_{\nu}\ell^{\nu} = 0\,, \tag{284}$$

meaning that the relevant symmetry generator ℓ^{μ} is tangential to the worldsheet, and the corresponding Lie invariance condition on its surface stress momentum energy density tensor will have the form

$$\ell^{\mu}\nabla_{\mu}\overline{T}^{\nu\rho} = 2\overline{T}^{\mu(\nu}\nabla_{\mu}\ell^{\rho)}\,. \tag{285}$$

Under such conditions, as well as the ordinary momentum flux $\overline{P}^{\mu}$, what may be termed the *adjoint momentum* flux,

$$^{\dagger}\overline{P}^{\mu} = \ell^{\nu}\,{}^{\dagger}\overline{T}_{\nu}{}^{\mu}\,, \qquad {}^{\dagger}\overline{T}_{\nu}{}^{\mu} = \mathcal{E}_{\nu\rho}\mathcal{E}^{\mu\sigma}\overline{T}^{\rho}{}_{\sigma}\,, \tag{286}$$

will also obey an equation of the form (279), i.e.

$$\overline{\nabla}^{\dagger}_{\mu}\overline{P}^{\mu} = \overline{f}_{\mu}\ell^{\mu}\,, \tag{287}$$

and hence would also be conserved if the string were free. In the presence of an electromagnetic or Kalb Ramond background, it can be seen that, like the ordinary momentum flux, this adjoint momentum flux has a gauge dependent extension,

$$^{\dagger}\overline{\mathcal{P}}^{\mu} = \ell^{\nu}\,{}^{\dagger}\overline{\mathcal{T}}_{\nu}{}^{\mu}\,, \qquad {}^{\dagger}\overline{T}_{\nu}{}^{\mu} = \mathcal{E}_{\nu\rho}\mathcal{E}^{\mu\sigma}\overline{\mathcal{T}}^{\rho}{}_{\sigma}\,, \tag{288}$$

that will share with the generalised momentum flux $\overline{\mathcal{P}}^{\mu}$ (that would be conserved even if the string did not share the symmetry of the background) the property of obeying a strict surface current conservation law, namely

$$\overline{\nabla}_{\mu}\,{}^{\dagger}\overline{\mathcal{P}}^{\mu} = 0\,. \tag{289}$$

The group invariance conditions

$$\ell^\nu \nabla_\nu p_\mu + p_\nu \nabla_\mu \ell^\nu = 0 \ , \qquad \ell^\nu \nabla_\nu \tilde{p}_\mu + \tilde{p}_\nu \nabla_\mu \ell^\nu = 0 \, , \tag{290}$$

that are the analogues of (285) for the separate mutually dual pair of internal momentum covectors $\tilde{p}_\mu$ and p_μ associated with the internal current within the string, can be rewritten, with the aid of the corresponding electromagnetic background invariance condition (280), in the form

$$\ell^\rho \mathcal{E}^{\mu\nu} \big(\nabla_\nu \tilde{p}_\mu + \frac{q}{2} F_{\nu\mu} \big) = \mathcal{E}^{\rho\nu} \nabla_\nu \mathcal{B} \, , \qquad \ell^\rho \mathcal{E}^{\mu\nu} \nabla_\nu p_\mu = \mathcal{E}^{\rho\nu} \nabla_\nu \beta \, , \tag{291}$$

in terms of a pair of scalars $\mathcal{B}$ and β, that can be considered as generalisations of the well known Bernoulli constant in stationary classical perfect fluid dynamics, and that are defined by

$$\mathcal{B} = \tilde{\beta} + q A_\mu \ell^\mu \ , \qquad \tilde{\beta} = \tilde{p}_\mu \ell^\mu \ , \qquad \beta = p_\mu \ell^\mu \ . \tag{292}$$

If ℓ^μ is timelike, the corresponding symmetry will be interpretable as *stationarity*, while the more restrictive case [103] in which the string is actually *static* (in the sense that there is no transverse current component relative to the background rest frame determined by ℓ^μ) will be given by the condition that the second Bernoulli constant, β should vanish.

Comparing the conditions (291) with (244) and (246) it can be seen that the *internal* dynamical equations are equivalent in this group invariant case to the corresponding pair of Bernoulli type conservation laws to the effect that $\mathcal{B}$ and β (but, unless the electromagnetic field is absent, not $\tilde{\beta}$) should both be *constant* over the worldsheet. This observation allows the problem of solving the dynamical for a symmetric configuration of the kind of ("perfectly elastic" , i.e. barotropic) string model under consideration to one of solving just the extrinsic equations governing the location of the worldsheet.

6.3 Generating tangent vector field for symmetric solutions

A recent investigation [104] based on the systematic use of variational methods in the restricted case for which the Kalb-Ramond coupling was absent has drawn attention to the interest of extrapolating the Bernoulli constants outside the supporting worldsheet as a pair of scalar fields defined over the entire background spacetime by the uniformity conditions

$$\nabla_\mu \mathcal{B} = 0 \, , \qquad \nabla_\mu \beta = 0 \, , \tag{293}$$

and formulating the problem in terms a certain preferred generating vector field that is constructed in such a manner as to be everywhere tangent to the worldsheet according to the specification

$$X^\mu = \mathcal{E}^{\mu\nu\dagger} \overline{P}_\nu = \ell^\nu \mathcal{E}_{\nu\rho} \overline{T}^{\rho\mu} \, . \tag{294}$$

Using (251) and (292), it can be seen that the generating vector defined in this way will be expressible in terms of the variables introduced in Section 5 as

$$X^\mu = \frac{\beta\Lambda}{\chi} c^\mu + \frac{\tilde\beta\tilde\Lambda}{\tilde\chi}\tilde c^\mu \,, \tag{295}$$

while the Killing vector itself will be expressible in analogous form by

$$\ell^\mu = \frac{\tilde\beta}{\Lambda - \tilde\Lambda}\, c^\mu + \frac{\beta}{\tilde\Lambda - \Lambda}\,\tilde c^\mu \,. \tag{296}$$

By further use of the formulae in Section 5 it can be seen that the squared amplitude of the latter (which in the case of stationary symmetry can be interpreted as an effective gravitational potential field) will be given by an expression of the form

$$\ell^2 = \ell^\mu \ell_\mu = -\frac{\beta^2}{\chi} - \frac{\tilde\beta^2}{\tilde\chi} \,. \tag{297}$$

To obtain a solution of the dynamical equations governing a symmetric string configuration of he kind under consideration it will evidently be sufficient to obtain an integral trajectory of the generating tangent vector X^μ, since when this is available the complete worldshhet can then be swept out in a trivial manner by dragging this trajectory along under the symmetry action generated by ℓ^μ. A very useful approach to the problem of finding sych a tangent trajectory is based on a procedure that exploits of the preceeding equation (297) in the following manner.

In view of (292), and of the state functional relationship (240) between χ and its dual, $\tilde\chi$, the equation (297) can be solved for any particular choice of the constant"tuning parameters" $\mathcal{B}$ and β to determine the internal variables χ and $\tilde\chi$ as functions of the (geometrically determined) potential ℓ^2 and the (electric type) potential $\ell^\nu A_\nu$, and hence by implication as scalar fields over the entire background space, not just on the worldsheet where they were originally defined.

Again using the formulae in Section 5, it can be seen from (295) and (296) that the contraction of the worldsheet generating vector X^μ with the Killing vector field will be given simply by

$$\ell^\mu X_\mu = \beta\tilde\beta \,, \tag{298}$$

and that the squared amplitude of X^μ will given by an expression of the form

$$X^2 = X^\mu X_\mu = \frac{\beta^2\Lambda^2}{\chi} + \frac{\tilde\beta^2\tilde\Lambda^2}{\tilde\chi} \,. \tag{299}$$

By the latter equation, $X^\mu X_\mu$ is also implicitly defined as a function of ℓ^2 and $\ell^\nu A_\nu$, so that it too can be considered defined scalar field, not just on the worldsheet but also over the background as a whole. The gradient of the field constructed in this way will be given by the expression

$$\nabla_\mu X^2 = -\Lambda\tilde\Lambda\,\nabla_\mu \ell^2 + 2q\tilde\beta(\Lambda - \tilde\Lambda)\frac{\tilde\Lambda}{\tilde\chi}\nabla_\mu(\ell^\nu A_\nu) \,. \tag{300}$$

Previous experience [104] with the case in which only the electromagnetic but not the Magnus force contribution is present suggests the interest of formulating the problem in terms of the propagation of the special generating vector X^μ, which (using the formulae (210) and (251) of the previous section) can be seen from (295) and (296) to be given by

$$X^\nu \nabla_\nu X_\mu = \Lambda\tilde{\Lambda}\, \ell^\nu \nabla_\nu \ell_\mu + X^\nu \mathcal{E}_{\nu\rho} \ell^\rho \overline{f}_\mu \,, \tag{301}$$

where $\overline{f}^\mu$ is the background force as given by (237). It can now be seen that the two preceeding equations can be combined to give the equation of motion for the worldsheet generating vector X^μ in the very elegant and convenient final form

$$X^\nu \nabla_\nu X_\mu - \tfrac{1}{2} \nabla_\mu X^2 = \mathcal{F}_{\mu\nu} X^\nu \ , \tag{302}$$

in terms of a pseudo Maxwellian field given by

$$\mathcal{F}_{\mu\nu} = q\beta F_{\mu\nu} + \overline{h} N_{\mu\nu\rho} \ell^\rho = 2\nabla_{[\nu} \mathcal{A}_{\rho]} \ , \tag{303}$$

where $\mathcal{A}_\mu$ is a gauge dependent pseudo-Maxwellian potential covector given by

$$\mathcal{A}_\mu = q\beta A_\mu + \overline{h} B_{\mu\nu} \ell^\nu \ . \tag{304}$$

If ℓ^μ is timelike so that the corresponding symmetry is interpretable as *stationarity*, then the equation of motion (302) will be interpretable as the condition for the string to be in *equilibrium* with the given values of the constant "tuning" parameters $\mathcal{B}$ and β, (of which, as remarked above, the latter, β, will vanish in the case of an equilibrium that is not just stationary but *static*, [103]). The new result here is that the Joukowski type "lift" force (which was not allowed for in the previous analysis [104]) due to the Magnus effect on the string, as it "flies" (like an aerofoil) through the background medium represented by the current 3-form $N_{\mu\nu\rho}$, has just the same form as an extra Lorentz type electromagnetic (indeed in the stationary case purely magnetic) force contribution.

6.4 Hamiltonian for world sheet generator

When the genuine electromagnetic background coupling, and the similarly acting Kalb Ramond coupling are both absent, then (as pointed out previously [104]) the equation of motion (302) (that is the equilibrium condition for "steady flight" in the stationary case) is just a simple geodesic equation with respect to, not the actual background spacetime metric $g_{\mu\nu}$, but the conformally modified metric $X^2 g_{\mu\nu}$, with the conformal factor X^2 determined as a field over the background by (299) in conjunction with (293) and (297). Even when the Lorentz and Joukowski force contributions are present, the equation (302) governing the propagation of the world sheet generator X^μ retains a particularly convenient Hamiltonian form, given by

$$X^\mu = \frac{dx^\mu}{d\tau} = \frac{\partial H}{\partial \Pi_\mu} \ , \qquad \frac{d\Pi_\mu}{d\tau} = -\frac{\partial H}{\partial x^\mu} \,, \tag{305}$$

for the quadratic Hamiltonian function

$$H = \tfrac{1}{2} g^{\mu\nu}(\mathit{\Pi}_\mu - \mathcal{A}_\mu)(\mathit{\Pi}_\nu - \mathcal{A}_\nu) - \tfrac{1}{2} X^2 \,, \tag{306}$$

subject to a restraint fixing the (generically non-affine) parametrisation τ of the trajectory by the condition that the numerical value of the Hamiltonian (which will automatically be a constant of the motion) should vanish,

$$H = 0 \,, \tag{307}$$

together with a further momentum restraint, determining the (automatically conserved) relative transport rate $\mathit{\Pi}_\mu \ell^\mu$ in accordance with the relation (298) by the condition

$$\mathit{\Pi}_\mu \ell^\mu = \mathcal{B}\beta \,. \tag{308}$$

The Hamiltonian momentum covector itself can be evaluated as

$$\mathit{\Pi}_\mu = X_\mu + \mathcal{A}_\mu = \ell^\nu \mathcal{E}_{\nu\rho} \overline{\mathcal{T}}^\rho{}_\mu \,. \tag{309}$$

In terms of the original conserved generalised momentum fluxes given by (283) and (288, and the gradients of the scalar (stream function and phase) potentials introduced in (230) and (247), this Hamiltonian covector will be given by

$$\mathit{\Pi}_\mu - \mathcal{A}_\nu \perp^\nu_\mu = \mathcal{E}_{\mu\nu}{}^\dagger \overline{\mathcal{P}}^\nu = \overline{\mathcal{P}}^\nu \mathcal{E}_{\nu\mu} + \mathcal{B}\overline{\nabla}_\mu \psi + \beta\sharp\overline{\nabla}_\mu \varphi \,. \tag{310}$$

(It is to be noted that there were omissions due to transcription errors in the analogues of the latter formulae in the preceeding version [1] of these notes.)

The advantage of a Hamiltonian formulation is that it allows the problem to be dealt with by obtaining the momentum covector in the form $\mathit{\Pi}_\mu = \nabla_\mu S$ from a solution of the corresponding Hamilton Jacobi equation, which in this case will take the form

$$g^{\mu\nu}\big(\nabla_\mu S - \mathcal{A}_\mu\big)\big(\nabla_\nu S - \mathcal{A}_\nu\big) = X^2 \,, \tag{311}$$

with X^2 given by (299) via (297), while the restraint (308) gives the condition

$$\ell^\nu \nabla_\nu S = \beta\mathcal{B} \,. \tag{312}$$

It is to be remarked that the square root X of X^2 may be purely imaginary since the generating vector may be timelike. Instead of working with X it has been found convenient [106] to work with a related field, Υ, that will be real if the Killing vector is spacelike so that ℓ is real, and that is given explicitly by either of the mutually dual (by (239) and (297) identically equivalent) alternative formulae

$$\Upsilon = \frac{\beta^2 \mathcal{K}}{\ell} - \tilde{\Lambda} = \frac{\tilde{\beta}^2 \tilde{\mathcal{K}}}{\ell} - \Lambda \ell \,. \tag{313}$$

Considered as a function of the field ℓ defined by (297) the derivative of Υ will be given by

$$\frac{d\Upsilon}{d\ell} = -\frac{\beta^2 \mathcal{K}}{\ell^2} - \Lambda = -\frac{\tilde{\beta}^2 \tilde{\mathcal{K}}}{\ell^2} - \tilde{\Lambda} \,, \tag{314}$$

which can be seen to be equivalent to

$$\ell^2 \frac{d\Upsilon}{d\ell} = \overline{T}^{\mu\nu} \ell_\mu \ell_\nu \tag{315}$$

It can be seen that the field X^2 in (311) will be expressible in terms of this new self-dual field $\Upsilon = \tilde{\Upsilon}$ in the form

$$X^2 = \frac{\beta^2 \tilde{\beta}^2}{\ell^2} - \Upsilon^2 \,. \tag{316}$$

6.5 Free evolution of circular loops

A particularly simple and important application of the formalism that has just been presented is to case [105], [106] of a small circular string loop undergoing motion that is free in the strong sense, meaning that there are no external forces of axionic, electromagnetic or even gravitational origin, i.e for which $B_{\mu\nu}$ and A_μ can be taken to vanish, and for which $g_{\mu\nu}$ can be taken to be flat, which will be a realistic approximation under a wide rang of circumstances, including cosmological applications in which the loop scale is small compared with the relevant Hubble length scale. There will be no loss of generality in taking such a circular loop to lie in the equatorial plane of a spherical coordinate system with origin at its center of symmetry, in terms of which the metric will have the standard form

$$g_{\mu\nu} dx^\mu dx^\nu = dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) - dt^2 \ . \tag{317}$$

This means that the loop lies in the hyperplane $\theta = \pi/2$, with a worldsheet whose internal coordinates could be taken to be t and ϕ. The extrinsic location of such a worldsheet will be specifiable just by giving its radius r as a function t. The Killing vector ℓ^μ generating the symmetry of the world sheet can be conveniently taken to be given by

$$\ell^\mu \frac{\partial}{\partial x^\mu} = 2\pi \frac{\partial}{\partial \phi} \,, \tag{318}$$

so that its magnitude ℓ as given by (297) will be directly identifiable with the circumference

$$\ell = 2\pi r \tag{319}$$

of the loop. The flux $\mathcal{J}^\mu$ of angular momentum as defined with respect to the usual angular momentum generator $\partial/\partial\phi$ will thus be given by the proportionality relation

$$\overline{P}^\mu = 2\pi \mathcal{J}^\mu \,, \tag{320}$$

where $\overline{P}^\mu$ is the momentum current (278) associated with ℓ^μ (which in this case is the same as that given by (283), since we are neglecting the effects of electromagnetic and axionic fields). The stationarity of the background (though

not of the worldsheet) with respect to the Killing vector k^μ that generates time translations according to the specification

$$k^\mu \frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial t}\,, \tag{321}$$

ensures that as well as the conserved angular momentum flux (320) there will also be a non-trivially conserved mass-energy flux vector that is specified by the corresponding analogue of (283) as

$$\mathcal{M}^\mu = -\overline{T}^\mu{}_\nu k^\nu\,. \tag{322}$$

For each of these independent conserved surface fluxes there will be a corresponding conserved constant characterising the loop, namely its total angular momentum

$$J = \oint dx^\mu\, \mathcal{E}_{\mu\nu} \mathcal{J}^\nu\,, \tag{323}$$

and its total mass

$$M = \oint dx^\mu\, \mathcal{E}_{\mu\nu} \mathcal{M}^\nu\,. \tag{324}$$

As well as these two quantities, whose conservation depends on the postulated summetry, there will be two other integrals that will be conserved in any case, namely the winding number N and the particle quantum number Z that are defined by (256) and (257), which can be seen to be determined in the present case by the Bernoulli constants (292), in terms of which they are given by

$$Z = \sharp\beta \qquad\qquad 2\pi\sharp N = \mathcal{B}\,. \tag{325}$$

The angular momentum and mass integrals can also be expressible by purely local formulae, in terms of the the momentum covector $\mathit{\Pi}_\mu$ for the worldsheet generator. Since the effective gauge field $\mathcal{A}_\mu$ vanishes in the present application this momentum covector will be given directly by the formula (294) as

$$\mathit{\Pi}_\mu = \ell^\nu \mathcal{E}_{\nu\rho} \overline{T}^\rho{}_\mu\,, \tag{326}$$

from which it can be seen that the corresponding total angular momentum and mass constants will be given by

$$\ell^\mu \mathit{\Pi}_\mu = 2\pi J\,, \qquad\qquad k^\mu \mathit{\Pi}_\mu = -M\,. \tag{327}$$

It is to be noticed that only three of the four conserved integrals J, M, Z, N are actually independent: by substituting (236) in (326) it can be seen that the first one is just the product of the last two, i.e.

$$J = ZN\,. \tag{328}$$

(It it to be remarked that yet another way of obtaining a conserved integral would to take the adjoint analogue ${}^\dagger J$ of the angular momentum that is constructed from ${}^\dagger\overline{P}^\mu$ as given by (286) instead of from $\overline{P}^\mu$ as given by (278)

according to the specification $2\pi\,{}^\dagger\!J = \ell^\mu \mathcal{E}_{\mu\nu}\,{}^\dagger\overline{P}^\nu$, but the result in this case is not independent: it can be seen that the angular momentum is self adjoint in the sense that ${}^\dagger\!J = J$.)

The mass-energy constant (327) is interpretable in the present case as the rate of variation of the ordinary coordinate time t with respect to the Hamiltonian parameter τ introduced in (305):

$$\frac{dt}{d\tau} = M\,. \tag{329}$$

Using a dot to denote differentiation with respect to the coordinate time t, the radial momentum component $\Pi_{_1}$ will correspondingly be given by

$$\Pi_{_1} = \frac{dr}{d\tau} = M\dot{r}\,. \tag{330}$$

In terms of this quantity the Hamiltonian (306) will be expressible as

$$H = \tfrac{1}{2}\big(\Pi_{_1}^{\,2} + \frac{J^2}{r^2} - M^2 - X^2\big)\,. \tag{331}$$

It is at this point that the identity (316) is useful, since it can be seen from (325) and (328) that it provides the convenient simplification

$$\frac{J^2}{r^2} - X^2 = \Upsilon^2\,. \tag{332}$$

The normalisation constraint (307) to the effect that the Hamiltonian should vanish thus provides the required equation of motion for the loop radius r in the conveniently first integrated form

$$M^2\dot{r}^2 = M^2 - \Upsilon^2\,, \tag{333}$$

where Υ is the function specified – independently of M – by the Bernoulli constants (i.e. by Z and N) according to the formula (313), whose explicit form depends of course on the equation of state.

For the case of a string model representing a Witten type superconducting vortex an appropriate approximation for the equation of state is provided by the formula (263) whose application to this problem has recently been the subject of detailed investigation [106] (following a more specialised study of the zero angular momentum case for which Z or N vanishes so that collapse is unavoidable [107]): the function Υ has no maximum, but for moderate values of the ratio of Z to N it has a finite minimum, and in that case, depending on the particular values of the three independent constants Z, N, and M the loop may either oscillate smoothly between a finite maximum and minimum value of the radius or else it may reach a critical minimum radius at which the model breaks down due to overconcentration of the current which results in a local instability: in the case of a timelike current such an instability arises because the "wiggle" speed $c_{_{\rm E}}$ given by (194) tends to zero, whereas in the case of a spacelike current it arises because the longitudinal characteristic speed $c_{_{\rm L}}$ given by (204) tends to zero.

6.6 Vorton equilibrium states

It can be seen from the final equation (333) of the Subsection 6.5 that the circular loop will have a stationary equilibrium state characterised by the minimum admissible value of M for given values of Z and N at the radius r for which Υ takes its minimum value. This happens for $M = \Upsilon$ where $d\Upsilon/dr$ vanishes, which according to (297) and (314) occurs where

$$\beta^2 \tilde{\Lambda} \tilde{\chi} + \tilde{\beta}^2 \Lambda \chi = 0 \,. \tag{334}$$

According to (315) this is equivalent to

$$\overline{T}^{\mu\nu} \ell_\mu \ell_\nu = 0 \tag{335}$$

which is interpretable as the condition that the stress in the direction of the axisymmetry generator should vanish.

Such centrifugally supported ring states are simple examples of the phenomenon for which Davis and Shellard [81], [82] have introduced the term "vorton". The possibility of forming "vortons", meaning centrifugally supported but not necessarily circular equilibrium states of cosmic string loops is of considerable potential importance for cosmology, but it was largely overlooked during the early years of cosmic string theory.

The formalism embodied in the generator equation (302) and the corresponding Hamiltonian (306) can be applied to the general study of stationary equilibrium states by replacing the axisymmetry vector ℓ to which it was applied in the preceeding subsection by the Killing vector k^μ that generates the stationary symmetry of the background, which, when substituted for ℓ^μ in (294), will determine a corresponding world sheet tangent vector

$$\sigma^\mu = k^\nu \mathcal{E}_{\nu\rho} \overline{T}^{\rho\mu} \,, \tag{336}$$

instead of the vector X^μ.

Such an approach provides results of considerable mathematical (as opposed to physical) interest when applied to the case of stationary strings in a black hole background. Generalising results obtained previously [108], [109] for the Goto Nambu limit case, it has been shown [104] that for a string model of the non-dispersive permanently trasonic type with the constant product equation of state (205) that is governed by the Lagrangian (267) (as obtained [5] both from the Nielsen dimensional reduction mechanism and also [89], as explained in Section 5, from the more physically realistic "wiggly" string approximation) the stationary Hamilton Jacobi equation is *exactly soluble by separation of variables* in a Kerr black hole spacetime, not just of the ordinary asymptotically flat kind but even of the generalised asymptotically De Sitter kind [110], [111]. Except in the Schwarschild-De Sitter limit, where it could of course have been predicted as a consequence of spherical symmetry, this separability property still seems rather miraculous, reflecting a "hidden symmetry" of the Kerr background that is still by no means well understood. The newly discovered separability property [104] is not just an automatic consequence of the simpler, though when first

discovered already surprising, property of separability for the ordinary geodesic equation [112] but depends on a more restrictive requirement of the kind needed for the more delicate separability property of the scalar wave equation [113]. (It is however more robust than the separability properties that have turned out to hold for higher spin bosonic [114] and fermionic [115], [116], [117] wave equations, and other related systems [118], [119].)

From a physical point of view the mathematically interesting configurations discussed in the preceeding paragraph are rather artificial. What is much more important from a cosmological point of view is the the equilibrium of *small closed string loops* in the mathematically relatively trivial case for which the background gravitational, electromagnetic, and axionic forces are *negligible*. In the absence of electromagnetic, or axionic forces, the relevant generator equation, which is obtained (by substituting σ^μ for X^μ in (302) with the source term on the right set to zero) will have the form

$$\sigma^\nu \nabla_\nu \sigma_\mu - \tfrac{1}{2} \nabla_\mu (\sigma_\nu \sigma^\nu) = 0 \,. \tag{337}$$

It is evident that the worldsheet generators obtained as solutions of this equation in a Minkowski background will all just be *straight lines* for the case of a stationarity Killing vector k, since in this case (unlike that of axisymmetry) $k^\nu k_\nu$ and hence also $\sigma^\nu \sigma_\nu$ will be constant. Since in this case the Killing vector trajectories are also straight, it might at first seem to follow that the worldsheet of a stationary string in an empty Minkowski background would necessarily be flat. This conclusion would exclude the possibility, when on background field is present, of any closed loop equilibrium states – such as the circular configurations characterised by (334). In fact cosmologists did indeed (albeit for other reasons) entirely overlook the possibility that such states might exist until the comparitively recent publication of an epoch making paper by Davis and Shellard [81] provided the first counterexamples (the only previously considered equilibrium states [120], [70], [71], [72] having been based on a magnetic support mechanism that was was finally judged to be too feeble to be effective except [122] as a minor correction).

The loophole in the deduction that if the trajectories generated by σ^μ and by k^μ are both straight then the worldsheet must be flat is that it is implicitly based on the assumption that the two kinds of trajectories cross each other transversly. However there will be no restriction on the curvature in the transverse direction in the critical case for which the two kinds of trajectory coincide, i.e. for which σ^μ and k^μ are parallel. The condition for criticality in this sense is expressible as

$$\sigma^\mu \mathcal{E}_{\mu\nu} k^\nu = 0 \,, \tag{338}$$

which can be seen from the definition (336) to be interpretable as meaning that the extrinsic characteristic equation (133) is satisfied by the tangent covector $\chi_\mu = \mathcal{E}_{\mu\nu} k^\nu$, which is proportional to ℓ_μ. This is equivalent to the condition that the Killing vector k^μ itself should be bicharacteristic, in the sense of being directed allong the propagation direction of extrinsic perturbations of the world sheet. The criticality condition (338) – which can be seen to be equivalent to

(334) – is thus interpretable as a condition of *characteristic flow*. It means that the "running velocity", v say, of relative motion of the intrinsicly preferred rest frame of the string (as determined by whichever of c^μ and $\tilde{c}^\mu$ is timelike) relative to the background frame specified by the (in this case necessarily timelike) Killing vector k^μ is the same as the extrinsic propagation velocity $c_{\rm E}$ given by (194), i.e.

$$v^2 = \frac{T}{U}\,. \tag{339}$$

In the presence of generic gravitational, electromagnetic and Kalb Ramond forces, the criticality condition (338) can be satisfied at particular positions, such as where there is a transition from a subcharacteristic running velocity, $v < c_{\rm E}$ to a supercharacteristic running velocity $v > c_{\rm E}$ (as will occur for instance on a string in a steady state of radial flow into a Schwarzschild black hole) or where there is a cusp, with $v = c_{\rm E}$ but with subcharacteristic flow $v < c_{\rm E}$ on both sides. However in view of the constancy of the Bernoulli "tuning" parameters $\mathcal{B}$ and β, the absence of any background field will always allow, and generically (the special integrable case (268) being an exception) will ensure, *uniformity* of the state of the string, so that the transcharacteristic flow condition (338) can be satisfied throughout its length. The space configuration of such a uniformly transcharacteristic steady string state can have arbitrarily variable curvature, and so is compatible with a closed loop topology.

The value of the mass M at which the equilibrium condition (339) is attained will of course be proportional to the circumference ℓ of the vorton: it can easily be seen that its mass to length ratio will be given by

$$\frac{M}{\ell} = U + T = -\Lambda - \tilde{\Lambda}\,. \tag{340}$$

For given values of the conserved quantum numbers N and Z, the absolute scale of the vorton will be given by the formula for its length which takes the comparatively simple form[83]

$$\ell^2 = \frac{2\pi|NZ|}{\sqrt{UT}} = \frac{2\pi|NZ|}{\sqrt{\Lambda\tilde{\Lambda}}}\,. \tag{341}$$

In the "chiral" case when the current is approximately null, and more generally in the magnetic case for which it is spacelike, one expects (see Subsection 5.7) the energy density and tension to be given by $U \approx T \approx m^2$, where m is the relevant Higgs mass scale. On this basis the two preceeding formulae are sufficient by themselves to provide reasonably accurate estimates of the vorton length and circumference. However if the current is timelike, U may become relatively large and T very small compared with m^2. To estimate their values in a case of this more general kind, it is necessary to solve the (non linear) equilibrium equation (334) which (since our assumption that electromagnetic effects are unimportant implies $\beta = \mathcal{B}$) determines the local state as a function of the ratio of the particle

number Z to the winding number N by a relation of the form

$$2\pi\sharp^2\mathcal{K}\sqrt{\frac{\tilde{\Lambda}}{\Lambda}} = \Big|\frac{Z}{N}\Big| . \tag{342}$$

6.7 *Allowance for the electromagnetic "spring" effect.

Instead of working with the generator tangent vector σ^μ that is given by the (not so obvious) specification (336), another way of deriving the conclusions of Subsection 6-6, and in particular the *centrifugal equilibrium* condition (339), is to use an approach whereby the quantities in the fundamental dynamical equation (122) are directly evaluated in terms of the worldsheet tangent vector e^μ that is given modulo a choice of sign by the (more obvious) specification that it be orthogonal to the Killing vector k^μ with unit normalisation,

$$e^\nu k_\nu = 0\ , \qquad e^\nu e_\nu = 1\, . \tag{343}$$

Since the Killing vector generator of ordinary stationary smmetry in flat space is covariantly constant and can be taken to have unit normalisation

$$\nabla_\mu k_\nu = 0\ , \qquad k^\nu k_\nu = -1\, , \tag{344}$$

the first fundamental of the stationary string worldsheet will be expressible by

$$\eta^{\mu\nu} = -k^\mu k^\nu + e^\mu e^\nu\ , \tag{345}$$

so that by (14) the second fundamental tensor will be obtainable simply as

$$K_{\mu\nu}{}^\rho = e_\mu e_\nu K^\rho\ , \tag{346}$$

where

$$K^\rho = e^\sigma \nabla_\sigma e^\rho\, . \tag{347}$$

Under these conditions, the preferred orthonormal diad introduced in Section 5.2 will be expressible in the form

$$u^\mu = (1-v^2)^{-1/2}\big(k^\mu + v e^\mu\big) \qquad v^\mu = (1-v^2)^{-1/2}\big(e^\mu + v k^\mu\big)\, , \tag{348}$$

where v is the longitudinal "running velocity" characterising the rate motion of relative to the stationary background of the internally preferred rest frame of the string that is determined generically by whichever of c^μ and $\tilde{c}^\mu$ is timelike. Using the expression (99) for the surface stress-energy tensor, the left hand side of the fundamental dynamical equation (122) can thus be evaluated as

$$\overline{T}^{\mu\nu} K_{\mu\nu}{}^\rho = \Big(\frac{Uv^2 - T}{1-v^2}\Big) K^\rho\, . \tag{349}$$

In the absence of external – electromagnetic or Magnus type – forces, the condition for equilibrium is that the vector given by (345) should vanish. This

requirement can be satisfied in one or other of two alternative manners. One way is for the configuration is to be *straight*, so that $K^\rho = 0$, in which case any value (including the *static* case $v = 0$) is possible for v. However a straight configuration is possible only if the string is infinite or else terminates at fixed endpoints. For a closed string loop in flat space the curvature vector K^ρ must necessarily be non-zero at least along part of the configuration, and in this case it is evident that the only way for the equilbrium condition to be satisfied is for the running velocity to satisfy the transcharacteristic condition (339), in which case there will be no restriction at all on the curvature vector K^ρ. Since by (194), the centrifugal equilibrium condition (339) is interpretable as meaning that the running velocity must be equal to the velocity of transverse ("wiggle") type perturbations, it can be seen to follow that relatively backward moving perturbations will have their flow speed exactly canceled out, i.e. they will just be static deformations, which explains why, in this transcharacteristic case the curvature K_ρ is unrestricted.

The formula (349) will of course still be applicable in more general cases, for which the dynamical equation (122) has a non vanishing right hand side expressing the effect of an electromagnetic or Magnus type force (the latter being responsible for the equilibrium of an ordinary "smoke ring", which is the most familiar laboratory example of a vorton type equilibrium state). In a cosmological context particular interest attaches to the case of electromagnetic self interaction arising in the case of a current with a non negligible electromagnetic charge coupling. Unlike the effect of a Magnus force or a purely external magnetic field which are straightforwardly tractable within the framework developed here, allowance for electromagnetic self interaction raises an awkward and delicate problem because the self induced field is singular on the worldsheet, so that its treatment (as in the more familiar case of a particle self interaction) requires the use of some sort of renormalisation procedure, typically involving a cut off length representing the finite thickness of the worldsheet in a microscopic description.

No satisfactory treatment of electromagnetic (or gravitational) self inetaction has yet been developed for the general, dynamically evolving case, but for the very special case of a stationary circular string loop an adequate analysis of the corrections arising from "spring" effect due to electromagnetic self interaction has been provided by Peter [84]. This treatment involves the introduction of a cut off length, r_*, say that represents the effective thickness of the microscopic current destribution in the string and that is expected to be of the order of magnitude of the Compton wavelength associated with the Witten carrier mass scale m_* that was introduced in Section 5.7, which (in Planck units) means $r_* \approx 1/m_*$. In the limit when this radius goes to zero the electromagnetic field is logarithmically divergent, its effective mean value within the string worldsheet being given in terms of the surface current vector $\overline{j}^\mu$ by an expression of the form

$$F_{\mu\nu} = -2 \ln\left\{\frac{r}{r_*}\right\}\left(K_{[\mu}k_{\nu]}k_\sigma + K_{[\mu}e_{\nu]}e_\sigma\right)\overline{j}^\sigma \,, \tag{350}$$

where r is the the radius of the circular configuration under consideration, which

will be given in terms of the corresponding curvature vector by

$$K^\rho K_\rho = \frac{1}{r^2}\,. \tag{351}$$

In terms of the number density current vector c^μ and the associated momentum vector p_μ as given by (230) and (234) the corresponding electromagnetic force vector will be given by

$$F_{\mu\nu}\overline{j}^\nu = -q^2 \ln\Big\{\frac{r}{r_*}\Big\}\big((c^\nu e_\nu)^2 + (p_\nu e^\nu)^2\big)K_\mu\,, \tag{352}$$

where q^2 is the effective charge coupling constant which will be given in terms of the charge per fundamental particle, $e^2 \simeq 1/137$ by $q^2 = \sharp^2 e^2$ where $\sharp$ is the normalisation factor introduced in Subsection 5.6, which is expected to be of order unity.

It is convenient to rewrite this in terms of the modulus $|\chi|$ of the squared current magnitude $\chi = c^\mu c_\mu$ of the current vector, which, according to the results described in Subsection 5.7, is expected to be subject to a saturation limit given in terms of the Witten mass scale m_* by $\chi \lesssim {m_*}^2$. Setting the resulting expression

$$F_{\mu\nu}\overline{j}^\nu = -q^2|\chi|\Big(\frac{1+v^2}{1-v^2}\Big)\ln\Big\{\frac{r}{r_*}\Big\}K_\mu\,, \tag{353}$$

equal to the term (349) in accordance with the fundamental dynamical equation (122), it can be seen that the curvature covector K_ρ cancels out again, as in the electromagnetically uncoupled case. The non-trivial equilibrium condition that remains is Peter's generalisation [84] of the preceeding formula (339) for the running velocity v, which is expressible in the form

$$v^2 = \frac{T-\epsilon^2}{U+\epsilon^2}\,, \tag{354}$$

where the correction term allowing for the electromagnetic self interaction is given by

$$\epsilon^2 = q^2|\chi|\ln\Big\{\frac{r}{r_*}\Big\}\,. \tag{355}$$

In any realistic cosmic string model of the kind described in Subsection 5.7 it is to be anticipated that we shall always have

$$\chi \lesssim {m_*}^2 \lesssim m^2 \lesssim U\,., \tag{356}$$

and one also expects to have

$$q^2 \approx e^2 \simeq 1/137\,. \tag{357}$$

More specifically, in the "magnetic" range, $\chi > 0$, for any such model, it is to be expected that the tension T will remain comparable with the energy density U, which implies that the condition

$$\epsilon^2 \ll T \approx m^2 \approx U \tag{358}$$

will be satisfied – unless the logarithmic factor is enormous, which is only conceivable for implausibly large values of the radius (too large for the ring to be treated as an isolated system unaffected by external matter). The electromagnetic force contribution will therefore be relatively negligible, not only in the "chiral" case, i.e. when current is approximately null, but also in the "magnetic" case considered in early work, i.e. when the current is spacelike. This means that the original notion [70], [71], [72] of non rotating magnetostatically supported "cosmic spring" configurations was unrealistic[77].

On the other hand however, as Peter has pointed out [84], if the definition of a "cosmic spring" is extended to include non-rotating configurations with *electrostatic* as opposed to magnetostatic support, then the preceeding objection no longer applies. This is because, unlike the spacelike case, the tension can become relatively small, $T \ll U$, in the case of a timelike current for a cosmic sting model of the kind described in Subsection 5.7. While the existence of exactly non rotating equilibrium states which, according to (354), would be characterised by the very special electrostatic support condition

$$T = \epsilon^2 \, . \tag{359}$$

will in principle be possible, nevertheless the formation of such "spring" type vortons would presumably be statistically rare under natural conditions. There will however be a more reasonable chance for vortons to be formed in the extended range of nearby slowly rotating states in which, if not actually dominant, electrostatic support does provide a contribution that may be non negligible compared with the purely centrifugal support mechanism considered Subsection 6.6. Although one would expect that such very slowly rotating states would initially represent only a small fraction of the total vorton population, they might come to be of dominant importance in the long run, due to the greater vulnerablity of supersonically rotating states to secular and even dynamical instability [78], [79], [80].

6.8 *Vortons in cosmology

The question of closed loop equilibrium states did not arise in the earliest studies [121] of cosmic strings, which were restricted to the Goto-Nambu model whose bicharacteristics are always null and so can never be aligned with the timelike Killing vector generating a stationary symmetry. However in a generic string model [36] for which the 2-dimensional longitudinal Lorentz of the internal structure is broken by a current, whether of the neutral kind exemplified in an ordinary violin string or the electromagnetic kind exemplified [97], [75], [76], [77] by Witten's superconducting vacuum vortex model, the bicharacteristic directions will generically be timelike [41] (spacelike bicharacteristics being forbidden by the requirement of causality) so there will be no obstacle to their alignment with a timelike Killing vector in accordance with the criticality condition (338), i.e. to having a running velocity given by $v = c_{\rm E}$. The simple "toy" complex scalar field model on which the pioneering cosmic string studies [6] were

based had longitudinally Lorentz invariant vortex defects (of "local" or "global" type depending on the presence or absence of coupling to a gauge field) that were describable at a microscopic level by string models (with Kalb Ramond coupling in the "global" case) that were indeed of the special Goto-Nambu type. However the extra degrees of field freedom (starting with the additional scalar field introduced by Witten in his original superconducting example [8]) that are needed in successively more realistic models [122], [123] make it increasingly difficult [124], [125] – though it may still be possible [126] – to avoid the formation of internal structure breaking the longitudinal Lorentz invariance and reducing the extrinsic characteristic velocity to the subluminal range $c_{\rm E} < 1$ at which stationary equilibrium with the critical running velocity $v = c_{\rm E} < 1$ becomes possible.

The cosmological significance of this is that whereas Goto Nambu string loops cannot ultimately avoid gravitationally or otherwise radiating away all their energy [127], [128], [129], since they have no equilibrium states into which they might settle down, on the other hand more general kinds of strings, whose occurrence would now seem at least as plausible [122], [123], [124], [125], can leave a relic distribution of stationary loop configurations that may survive indefinitely [81], [82], [130], [131], [132], [133], [134]. One would expect such configurations to be those that minimise the energy for given values of the relevant globally conserved quantities, of which there might be a considerable number in the more complicated multiply conducting models[94] that might be considered, but of which there are only a single pair in the "barotropic" type string considered here, namely the quantities Z and N defined in Subsection 5.6, which are proportional to the stream function winding number $\oint d\psi$ and the phase winding number $\oint d\varphi$, whose respective constancy results from the conservation of the mutually dual pair of currents c^μ and $\tilde c^\mu$. If the equilibrium is predominantly due to a single current with corresponding particle and winding numbers Z and N, then according to the formulae (340) and (341), the corresponding vorton mass M and length ℓ will have an order of magnitude given in terms of the relevant Higg's mass scale m by rough estimates of the form

$$M \approx |ZN|^{1/2} m \,, \qquad \ell \approx |ZN|^{1/2} m^{-1} \,, \tag{360}$$

whenever the current is spacelike (the "magnetic" case), or approximately null (the "chiral" case), so that one has $T \approx U \approx m^2$. The exceptions – for which considerably higher values may be obtained – are very slowly rotating low tension states, with a strong timelike current, which according to (342) will occur only for $Z^2 \gg N^2$: this would presumably be comparatively rare on the basis of random statistical fluctuations, though it might ultimately be favoured by selection if, as is not implausible, the initially more common (supersonically rotating) states with Z^2 comparable to N^2, turn out to be relatively unstable in the long run.

The potential cosmological importance of such a distribution of stationary relic loops, referred to as "vortons", was first pointed out by Davis and Shellard [81], [82] [130], who emphasised that in the case of the "heavyweight" cosmic

strings whose formation during G.U.T. symmetry breaking had been postulated to account for galaxy formation, the density of the ensuing "vortons" would be more that sufficient to give rise to a catastrophic cosmological mass excess (of the kind first envisaged as arising from the formation of monopoles) even if they were formed with very low efficiency. This prediction was implicitly based on the assumption that the Witten carrier mass scale, represented by the parameter m_* in (263), was of the same order of magnitude as the relevant Kibble mass scale, m, determined by the Higgs field, as would be the case if both the string formation and the current condensation were part of the same G.U.T. symmetry breaking phase transition, for which the corresponding gravitational coupling constant would be given by $\sqrt{Gm^2} \approx 10^{-3}$.

Subject to the foregoing assumption – i.e. that the Witten mass scale m_* was not much smaller than the Kibble mass scale m – early quantitative estimates suggested [131], [132] that, to avoid a cosmological mass excess if the vortons were formed with relatively high efficiency, the relevant phase transition energy scale could not have been much greater than a limit given very roughly by $\sqrt{Gm^2} \approx 10^{-13}$, which is not so very much larger than the scale of electroweak unification. This limit has however been subject to upward revision by more recent and detailed investigations, including allowance for more realistic – i.e. lower – estimates of the efficiency of vorton formation [133], [134]; the limit is further relaxed if one drops the supposition that the Witten mass m_* should be close to the Kibble mass scale m, so that provided $\sqrt{G{m_*}^2} \gtrsim 10^{-10}$ string formation at the G.U.T. scale $\sqrt{Gm^2} \approx 10^{-3}$ is cosmologically admissible after all even if the ensuing vortons are stable enough to survive to the present epoch. If the vortons are only stable enough to last for a few minutes, the condition that the standard nucleosynthesis process should not have been disrupted provides a somewhat weaker limit given [134] roughly by $\sqrt{G{m_*}^2} \gtrsim 10^{-12}$. It follows from this work that if the current condensation does not occur until the epoch of electroweak symmetry breaking then the ensuing vortons would contribute at most a very small fraction of the present cosmological closure density. This conclusion applies in particular to scenarios in which the relevant strings are themselves formed during electroweak symmetry breaking – which would happen only in a non-standard (e.g. supersymmetric) theory, since the standard Glashow Weinberg Salam version, does not give rise to stable string like vortex defects [135]. However even if their density is low by cosmological standards it does not necessarily follow that the vortons would be undetectable: for example it has recently been suggested [136] that they might account for otherwise inexplicable cosmic ray observations.

The potential importance of such cosmological and astrophysical effects provides the motivation for more thorough investigation of equilibrium states that may be involved, a particularly important question being that of their stability. Prior to the derivation of the general symmetric string generator equation (302), the only closed loop equilibrium states to have been considered were the circular kind to which the use by Davis and Shellard [81], [130] of the term "vorton" was originally restricted. The first general investigation [5], [83] of such circular

"cosmic ring" states showed that under conditions of purely centrifugal support (neglecting possible electromagnetic corrections of the kind evaluated more recently [84]) the condition (339) for equilibrium, namely the requirement of a transcharacteristic rotation speed $v = c_{\rm E}$, is such that the ring energy is minimised with respect to perturbations preserving the circular symmetry. However a more recent investigation of non-axisymmetric perturbations has shown that although there are no unstable modes for states of subsonic rotation [78] (as exemplified by a cowboy's lassoe loop) with $v = c_{\rm E} < c_{\rm L}$, instability can nevertheless occur for rotation in the supersonic regime [79], [80] that (contrary to what was implicitly assumed in earlier work [68], [69], [70], [71], [72], [73], [74] using the subsonic type of model given by a linear equation of state for which the sum $U + T$ is constant) has been shown by Peter [75], [76], [77] to be relevant in the kind of cosmic vortex defects that have been considered so far. Although the first category of string loop equilibrium states to have been studied systematically has been that of circular ring configurations [83], it has been made clear by recent work [104], [78] that, as explained above, arbitrary non circular equilibrium states are also possible. The stability of such more general equilibrium states has not yet been investigated. While it seems plausible that some kinds of "vorton" relic loops may be destroyed by the recently discovered classical instability mechanism [78], [79], and also the kind of quantum tunnelling instability mechanism considered by Davis [130], it does not seem likely that such mechanisms could be so consistently efficient as to prevent the long term survival of a lot of other "vorton" equilibrium states.

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References

[1] B. Carter, "Dynamics of cosmic strings and other brane models", in *Formation and Interactions of Topological Defects* (NATO ASI **B349**), ed. R. Brandenberger, A.-C. Davis, *pp* 304-348 (Plenum, New York, 1995).

[2] P.A.M. Dirac, "An extensible model of the electron", *Proc. Roy. Soc. Lond.* **A268**, *pp* 57-67 (1962).

[3] P.S. Howe, R.W. Tucker, "A locally supersymmetric and reparametrisation invariant action for a spinning membrane", *J. Phys.* **A10**, i pp L155-62 (1977).

[4] A. Achúcarro, J. Evans, P.K. Townsend, D.L. Wiltshire, "Super *p*-branes", *Phys. Lett.* **198 B**, *pp* 441-446 (1987).

[5] B. Carter, "Covariant mechanics of simple and conducting cosmic strings and membranes", in *Formation and Evolution of Cosmic Strings,* ed. G. Gibbons, S. Hawking, T. Vachaspati, *pp* 143-178 (Cambridge U.P., 1990).

[6] T.W.B. Kibble, "Topology of cosmic domains and strings", *J. Phys.* **A9**, *pp* 1387-98 (1976).

[7] A. Vilenkin, A.E. Everett, "Cosmic strings and domain walls in models with Goldstone and pseudo-Goldstone bosons", *Phys. Rev. Lett.* **48**, *pp* 1867-70 (1982).

[8] E. Witten, "Superconducting strings", *Nucl. Phys.* **B249**, *pp* 557-592 (1985).

[9] B. Carter, P. Peter, *Phys. Rev.* **D52**, *pp* 1744-48 (1995).

[10] B. Carter, "Outer curvature and conformal geometry of an imbedding", *J. Geom. Phys.* **8**, *pp* 53-88 (1992).

[11] B. Carter, "Basic brane theory", *J. Class. Quantum Grav.* **9**, *pp* 19-33 (1992).

[12] L.P. Eisenhart, *Riemannian Geometry* (Princeton U.P., 1926, reprinted 1960).

[13] J. Stachel, "Thickenning the string: the perfect string dust", Phys. Rev. **D21**, *pp* 2171-81 (1980).

[14] J.A. Schouten, *Ricci Calculus* (Springer, Heidelberg, 1954).

[15] S.W. Hawking, G.F.R. Ellis, *The Large Scale Structure of Spacetime* (Cambridge U.P., 1973).

[16] B. Carter, D. Langlois, "Kalb-Ramond coupled vortex fibration model for relativistic fluid dynamics", *Nucl. Phys.* **B454**, *pp* 402-424 (1995) [hep-th/9611082].

[17] Capovilla, R., and Guven, J., "Large deformations of relativistic membranes: a generalistion of the Raychaudhuri equations", *Phys. Rev.* **D52**, 1072 (1995) [gr-qc/9411061].

[18] R. Penrose, W. Rindler, *Spinors and Space-Time* (Cambridge U.P., 1984).

[19] I. Bars, C.N. Pope, *Class. Quantum Grav.,* **5**, 1157 (1988).

[20] P. Sikivie, "Axions, domain walls, and the early universe", *Phys. Rev. Lett.* **48**, *pp* 1156-59 (1982).

[21] E.P.S. Shellard, "Axion strings and domain walls", in *Formation and Evolution of Cosmic Strings,* ed. G. Gibbons, S. Hawking, T. Vachaspati, *pp* 107-115 (Cambridge U.P., 1990).

[22] Y. Nambu, "String-like configurations in the Weinberg-Salam theory", *Nucl. Phys.* **B130**, *pp* 505-515 (1977).

[23] N.S. Manton, "Topology in the Weinberg-Salam theory", *Phys. Rev.* **D28**, *pp* 2019-26 (1983).

[24] E. Copeland, D. Haws, T.W.B. Kibble. D. Mitchel, N. Turok, "Monopoles connected by strings", *Nucl. Phys.* **B298**, *pp* 458-492 (1988).

[25] T. Vachaspati, A. Achúcarro, "Semilocal cosmic strings", *Phys. Rev.* **D44**, *pp* 3067-71 (1991).

[26] T. Vachaspati, M. Barriola, "A new class of defects", *Phys. Rev. Lett.* **69**, *pp* 1867-72 (1992).

[27] X. Martin, A. Vilenkin, "Gravitational background from hybrid topological defects" *Phys. Rev. Lett.* **77**, 2879 (1996) [astro-ph/9606022].

[28] A. Dabholkar, J.M. Quashnock, "Pinning down the axion", *Nucl. Phys.* **B333**, *pp* 815-832 (1990).

[29] R.A. Battye, E.P.S. Shellard, "String radiative backradiation", *Phys. Rev. Lett.* **75**, (1995) [astro-ph/9408078].

[30] R.A. Battye, E.P.S. Shellard, "Radiative backreaction on global strings", *Phys. Rev.* **D53**, 1811 (1996) [hep-ph/9508301].

[31] B. Carter, "Axionic vorticity variational formulation for relativistic perfect fluids", *Class. Quantum Grav.* **11**, *pp* 2013-30 (1994).

[32] A. Vilenkin, T. Vachaspati, "Radiation of Goldstone bosons from cosmic strings" *Phys. Rev.* **D35**, *pp* 1138-40 (1987).

[33] R.L. Davis, E.P.S. Shellard, "Global strings and superfluid vortices", *Phys. Rev. Lett.* **63**, *pp* 2021-24 (1989).

[34] M. Sakellariadou, "Radiation of Nambu-Goldstone bosons from infinitely long strings", *Phys. Rev.* **D44**, *pp* 3767-73 (1991).

[35] U. Ben-Ya'acov, "Unified dynamics of quantum vortices", *Nucl. Phys.* **B382**, *pp* 597-615 (1992).

[36] B. Carter, "Duality relation between charged elastic strings and superconducting cosmic strings", *Phys. Lett.* **B224**, *pp* 61-66 (1989).

[37] A.L. Larsen, "A note on dispersive versus non-dispersive strings", *Class. Quantum Grav.* **10**, *pp* L35-38 (1993).

[38] A. Vilenkin, "Cosmic string dynamics with friction", *Phys. Rev.* **D43**, *pp* 1060-62 (1991).

[39] J. Garriga, M. Sakellariadou, "Effects of friction on cosmic strings", *Phys. Rev.* **48**, *pp* 2502-15 (1993) [gr-qc/9307008].

[40] B. Carter, M. Sakellariadou, X. Martin, "Cosmological expansion and thermodynamic mechanisms in cosmic string dynamics", *Phys. Rev.* **D50**, *pp* 682-99 (1994).

[41] B. Carter, "Stability and characteristic propagation speeds in superconducting cosmic and other string models", *Phys. Lett.* **B228**, *pp* 466-470 (1989).

[42] A.H. Taub, *Commun. Math. Phys.* **15**, 235 (1965).

[43] J.L. Friedman, B.F. Schutz, "On the stability of relativistic systems", *Astroph. J.* **200**, *pp* 204-220 (1975).

[44] B. Carter, "Perturbation dynamics for membranes and strings governed by Dirac Goto Nambu action in curved space", *Phys. Rev.* **D48**, *pp* 4835-38 (1993)

[45] J. Guven, "Perturbations of a topological defect as a theory of scalar fields interacting with an external vector potential", *Phys. Rev.* **D48**, *pp* 5563-69 (1993).

[46] A.L. Larsen, V. Frolov, "Propagation of perturbations along strings" *Nucl. Phys.* **B414**, 129 (1994) [hep-th/9303001].

[47] J. Garriga, A. Vilenkin, "Perturbations of domain walls and strings: a covariant theory" *Phys. Rev.* **D44**, *pp* 1007-14 (1991).

[48] J. Garriga, A. Vilenkin, "Quantum fluctuations on domain walls, strings, and vacuum bubbles", *Phys. Rev.* **D45**, *pp* 3469-86 (1992).

[49] J. Garriga, A. Vilenkin, "Black holes from nucleating strings", *Phys. Rev.* **D47**, *pp* 3265-74 (1993).

[50] J. Guven, "Covariant perturbations of domain walls in curved spacetime", *Phys. Rev.* **D48**, *pp* 4604-08 (1993).

[51] B. Carter, "Rheometric structure theory, convective differentiation, and continuum electrodynamics", *Proc. Roy. Soc. Lond.* **A372**, *pp* 169-200 (1980).

[52] A. Polyakov, "Fine structure on strings", *Nucl. Phys.* **B268**, *pp* 406-412 (1986).

[53] K.I. Maeda, N. Turok, "Finite width corrections to the Nambu action for the Nielsen-Olesen string", *Phys. Lett.* **B202**, *pp* 376-84 (1988).

[54] R. Gregory, "The effective action for a cosmic string", *Phys. Lett.* **B206**, *pp* 199-204 (1988).

[55] R. Gregory, "Effective actions for bosonic topological defects", *Phys. Rev.* **D43**, *pp* 520-25 (1993).

[56] D. Garfinkle, R. Gregory, "Corrections to the thin wall approximation in General Relativity", *Phys. Rev.* **D41** *pp* 1889-94 (1990).

[57] R. Gregory, D. Haws, D. Garfinkle, "Dynamics of domain walls and strings". *Phys. Rev.* **D42** *pp* 343-345 (1991).

[58] V. Silveira, M.D. Maia, "Topological defects and corections to the Nambu action", *Phys. Lett.* **A174**, *pp* 280-288 (1993).

[59] C. Barrabès, B. Boisseau, M. Sakellariadou, "Gravitational effects on domain walls with curvature corrections", *Phys. Rev.* **D49**, *pp* 2734-39 (1994).

[60] B. Carter, R. Gregory, "Curvature corrections to dynamics of domain walls", *Phys. Rev.* **D51**, *pp* 5839-46 (1995) [hep-th/9410095].

[61] P.S. Letelier, "Nambu bubbles with curvature corrections", *Phys. Rev.* **D41**, *pp* 1333-35 (1990).

[62] D. H. Hartley, R.W. Tucker, in *Geometry of Low Dimensional Manifolds, 1* (L.M.S. Lecture Note Series **150**, ed. S. Donaldson, C. Thomas (Cambridge U.P., 1990).

[63] H. Arodz, A. Sitarz, P. Wegrzyn, *Act. Phys. Polon. B*, **22**, 495 (1991); **23**, 53 (1992).

[64] B. Boisseau, P.S. Letelier, "Cosmic strings with curvature corrections", *Phys. Rev.* **D46**, *pp* 1721-29 (1992).

[65] B. Carter, "Equations of motion of a stiff geodynamic string or higher brane", *Class. Quantum Gravity* **11**, *pp* 2677-92 (1994).

[66] R. Capovilla, J. Guven, "Geomety of deformations of relativistic membranes", *Phys. Rev.* **D51**, 6736 (1995) [gr-qc/9411060].

[67] B. Carter, "Perfect fluid and magnetic field conservation laws in the theory of black hole accretion rings", in *Active Galactic Nuclei*, eds. C. Hazard, S.Mitton, pp 273-300 (Cambridge U. P., 1976).

[68] D.N. Spergel, T. Piran, J. Goodman, "Dynamics of superconducting cosmic strings", *Nucl. Phys.* **B291**, *pp* 847-875 (1987).

[69] A. Vilenkin, T. Vachaspati, "Electromagnetic radiation from superconducting cosmic strings", *Phys. Rev. Lett.* **58**, *pp* 1041-44 (1987)

[70] E. Copeland, M. Hindmarsh, N. Turok, "Dynamics of superconducting cosmic strings", *Phys. Rev. Lett.* **58**, *pp* 1910-12 (1987)

[71] E. Copeland, D. Haws, M. Hindmarsh, N. Turok, "Dynamics and radiation from superconducting strings and springs", *Nucl. Phys.* **B306**, *pp* 908-930 (1988).

[72] D. Haws, M. Hindmarsh, N. Turok, "Superconducting strings or springs?", *Phys. Lett.* **B209**, *pp* 255-261 (1988).

[73] D.N. Spergel, W.H. Press, R.J. Scherrer, "Electromagnetic self interaction of superconducting cosmic strings", *Phys. Rev.* **D39**, *pp* 379-388 (1989)

[74] P. Amsterdamski, "Evolution of superconducting cosmic loops", *Phys Rev.* **D39**, *pp* 1524-36 (1989).

[75] P. Peter, "Superconducting cosmic string: equation of state for spacelike and timelike currrent in the neutral limit", *Phys. Rev.* **D45**, *pp* 1091-102 (1992).

[76] P. Peter, "Influence of electric coupling strength in current carrying cosmic string", *Phys. Rev.* **D46**, *pp* 3335-49 (1992).

[77] P. Peter, "No cosmic spring conjecture", *Phys. Rev.* **D47**, *pp* 3169-76 (1993).

[78] B. Carter, X. Martin, "Dynamic instability criterion for circular string loops", *Ann. Phys.* **227**, *pp* 151-171 (1993).

[79] X. Martin, "Zones of dynamical instability for rotating cosmic string loops", *Phys. Rev.* **D50**, *pp* 7479-92 (1994).

[80] X. Martin, P. Peter, "Dynamical stability of Witten rings", *Phys. Rev.* **D51**, 4092 (1995) [hep-ph/9405220].

[81] R.L. Davis, E.P.S. Shellard, "The physics of vortex superconductivity", *Phys. Lett* **B209**, *pp* 485-490 (1988).

[82] R.L. Davis, E.P.S. Shellard, "Cosmic vortons", *Nucl. Phys.* **B323** *pp* 209-2024 (1989)

[83] B. Carter, "Mechanics of cosmic rings", *Phys. Lett.* **B238**, *pp* 166-171 (1990).

[84] P. Peter, "Electromagnetically supported string loops", *Phys. Lett.* **B298**, 60-62 (1993).

[85] N.K. Nielsen, "Dimensional reduction and classical strings", *Nucl. Phys.* **B167**, *pp* 249-260 (1980).

[86] N.K. Nielsen, P. Olesen, "Dynamical properties of superconducting cosmic strings", *Nucl. Phys.* **B291**, *pp* 829-846 (1987).

[87] A. Davidson, A.K. Wali, "Abelian self interaction of a test string", *Phys. Lett.* **B213**, *pp* 439-442 (1988).

[88] A. Davidson, A.K. Wali, “Non-Abelian self interaction of a test loop”, *Phys. Rev. Lett.* **61**, *pp* 1450-56 (1988).

[89] B. Carter, “Integrable equation of state for noisy cosmic string”, *Phys. Rev.* **D41**, *pp* 3869-72 (1990).

[90] A. Vilenkin, “Effect of small scale structure on dynamics of cosmic string”, *Phys. Rev.* **D41**, *pp* 3038-40 (1990).

[91] J.Hong, J.Kim, P. Sikivie, “Wiggly relativistic strings”, *Phys. Rev. Lett.* **69**, *pp* 2611-14 (1992).

[92] B. Carter, “Transonic elastic model for wiggly Goto Nambu string”, *Phys. Rev. Lett.* **74**, *pp* 3098-3101 (1995) [hep-th9411231].

[93] X. Martin, “Cancellation of longitudinal contribution in wiggly string equation of state”, *Phys. Rev. Lett.* **74**, *pp* 3102-3104 (1995).

[94] B. Carter, “Cold, warm, and composite (cool) cosmic string models”, *Nucl. Phys.* **B412**, *pp* 345-371 (1994).

[95] E. Witten, “Cosmic superstrings”, *Phys. Lett.* **B 153**, *pp* 243-246 (1985).

[96] R.L. Davis, E.P.S. Shellard, “Antisymmetric tensors and spontaneous symmetry breaking”, *Phys. Lett.* **B214**, *pp* 219-223 (1988).

[97] A. Babul, T. Piran, D.N. Spergel, “Bosonic superconducting cosmic strings: classical field theory solutions”, *Phys. Lett.* **B202**, *pp* 307-314 (1988).

[98] T.W.B. Kibble, N. Turok, “Self intersection of cosmic strings”, *Phys. Lett.* **B116**, *pp* 141-143 (1982).

[99] B. Allen, E.P.S. Shellard, “Cosmic string evolution: a numerical simulation”, *Phys. Rev. Lett.* **64**, *pp* 119-122 (1990).

[100] E.P.S. Shellard, B. Allen, “On the evolution of cosmic strings”, in *Formation and Evolution of Cosmic Strings,* ed. G. Gibbons, S. Hawking, T. Vachaspati, *pp* 421-448 (Cambridge U.P., 1990).

[101] A.G. Smith, A. Vilenkin, ‘Numerical simulation of cosmic-string evolution in flat spacetime”, *Phys. Rev.* **D36**, *pp* 990-996 (1987).

[102] A. Albrecht, “Small scale structure on cosmic strings”, in *Formation and Evolution of Cosmic Strings,* ed. G. Gibbons, S. Hawking, T. Vachaspati, *pp* 403-419 (Cambridge U.P., 1990).

[103] B. Carter, “Tuned String Geometries for Stationary Gravitational Fields”, *Class. and Quantum Grav.* **7**, *pp* L69-74 (1990).

[104] B. Carter, V.P. Frolov, O. Heinrich, "Mechanics of Stationary Strings: Separability for non-dispersive models in black hole background" *Class. and Quantum Grav.* **8**, *pp* 135-161 (1991).

[105] A.L. Larsen, "Dynamics of cosmic strings and springs: a covariant formulation", *Class. Quantum Grav.* **10**, *pp* 1541-48 (1993).

[106] B. Carter, P. Peter, A. Gangui, "Avoidance of collapse by circular current-carrying cosmic string loops", *preprint* (SISSA, 1996) [hep-ph/9609401].

[107] A.L. Larsen, M. Axinedes, "Runaway collapse of Witten vortex loops", *Class. Quantum Grav.* **14**, *pp* 443-453 (1997). [hep-th/9604135]

[108] V.P. Frolov, V.D. Skarzhinsy, A.I. Zelnikov, O. Heinrich, "Equilibrium configurations of a cosmic string near a rotating black hole", *Phys. Lett.* **224**, *pp* 255-258 (1989).

[109] B. Carter, V.P. Frolov, "Separability of String Equilibrium Equations in Generalised Kerr - De Sitter Background", *Class. Quantum Grav.* **6**, *pp* 569-580 (1989).

[110] B. Carter, "Black Hole Equilibrium States: I Analytic and Geometric Properties of the Kerr Solutions", in *Black Holes (Les Houches, 1972)*, ed. C. and B.S. DeWitt, *pp* 57-124 (Gordon and Breach, New York, 1973).

[111] G.W. Gibbons, S.W. Hawking, "Cosmologica event horizons, thermodynamics, and particle creation", *Phys. Rev.* **D15**, *pp* 2738-51 (1976).

[112] B. Carter, "Global Structure of the Kerr Family of Gravitational Fields", *Phys. Rev.* **174**, 1559 (1968).

[113] B. Carter, "Hamilton-Jacobi and Schrödinger Separable Solutions of Einstein's Equations", *Commun. Math. Phys.* **10**, 280 (1968).

[114] S.A. Teukolsky, "Perturbations of a rotating black hole", *Astroph. J.* **185**, *pp* 635-647 (1973).

[115] W. Unruh, "Separability of the neutrino equation in a Kerr background" *Phys. Rev. Lett.* **31**, *pp* 1265-67 (1973).

[116] S. Chandrasekhar, "The solution of Dirac's equation in Kerr geometry", *Proc. Roy. Soc. Lond.* **A349**, *pp* 571-575 (1976).

[117] R. Guven, "Black holes have no superhair", *Phys. Rev.* **D22**, *pp* 2327-30 (1980).

[118] J-A. Marck, "Solution to the equations of parallel transport in Kerr geometry; tidal tensor", *Proc. Roy. Soc. Lond.* **A385**, *pp* 431-438 (1983).

[119] B. Carter, "Separability of the Killing-Maxwell system underlying the generalised angular momentum constant in the Kerr-Newman black hole metrics", *J. Math. Phys.* **28**, *pp* 1535-38 (1987).

[120] J. Ostriker, C. Thompson, E. Witten, "Cosmological effects of superconducting strings", *Phys. Lett.* **B180**, *pp* 231-239 (1986)

[121] T.W.B. Kibble, "Some implications of a cosmological phase transition", *Phys. Rep.* **67**, *pp* 183-199 (1980).

[122] P. Peter, "Low mass current-carrying cosmic string", *Phys. Rev.*, **D46**, *pp* 3322-34 (1992).

[123] W. B. Perkins, A-C. Davis, "Cosmic strings and electroweak symmetry restoration", *Nucl. Phys.* **B406**, *pp* 377-408 (1993).

[124] P. Peter, "Spontaneous current generation in cosmic strings", *Phys. Rev.* **D49**, *pp* 5052-62 (1994) [hep-ph/9312280].

[125] A-C. Davis., P. Peter, "Cosmic strings are current carrying", *Phys. Lett.* **B358**, *pp* 197-202 (1995) [hep-ph/9506433].

[126] M. Goodband, M. Hindmarsh, "Cosmic electroweak strings", *Phys. Lett.* **B370**, 29 (1966) [hep-ph/9510434].

[127] N. Turok, "Grand unified strings and galaxy formation", *Nucl. Phys.* **B242**, *pp* 520-541 (1984).

[128] T. Vachaspati, A. Vilenkin, "Gravitational radiation from cosmic string", *Phys Rev.* **D31**, *pp* 3052-58 (1985).

[129] R. Durrer, "Gravitational angularmomentum radiation from cosmic strings", *Nucl. Phys.* **B328**, *pp* 238-271 (1989).

[130] R.L. Davis, "Semitopological solitons", *Phys. Rev.* **D38**, *pp* 3722-30 (1988).

[131] B. Carter, "Cosmic Rings as a Chump Dark Matter Candidate?", in *Early Universe and Cosmic Structures* (Xth Moriand Astrophysics meeting), ed. J.-M. Alimi, A. Blanchard, F. Martin de Volnay & J. Tran Than Van, *pp* 213-221 (Editions Frontières, Gif-sur-Yvette, 1990) .

[132] B. Carter, "Cosmological Relic Distribution of Conducting String Loops", *Ann. N.Y. Acad. Sci.* **647**, *pp* 758-766 (1991).

[133] B. Carter, in *Dark Matter in Cosmology* (XXXth Rencontre de Moriand, XVth Moriand Workshop), ed. B. Guiderdoni, G. Greene, D. Hinds & J. Tran Thanh Van, *pp* 195-204 (Editions Frontières, Gif-sur-Yvette, 1995) [hep-ph/9511225] .

[134] R. Brandenberger, B. Carter, A-C. Davis & M. Trodden, "Cosmic vortons and particle physics constraints", *Phys.Rev.* **D54**, *pp* 6059-6071 (1966) [hep-ph/9605382].

[135] M. James, L. Perivolaropoulos, T. Vachaspati, "Detailed stability analysis of electroweak strings", *Nucl. Phys.* **B395**, 534 (1993)[hep-ph/9212301].

[136] S. Bonazzola, P. Peter, "Can high energy cosmic rays be vortons?", *preprint*, (Obs. Paris-Meudon, 1997) [hep-ph/9701246].